One gets

$$k\cdot\sigma k\cdot\bar{\sigma}=m^2,$$

$(k\cdot\sigma)^{1/2}$ is the Hermitian square root of $k\cdot\sigma$.

The relation between the unimodular matrices and the restricted Lorentz transformations is given by

$$A\sigma_\mu A^\dagger=\Lambda_\mu{}^\nu\sigma_\nu,$$

or

$$\Lambda^\mu{}_\nu(\pm A)=\tfrac{1}{2}\,\mathrm{Tr}(\bar{\sigma}^\mu A\sigma_\nu A^\dagger).$$

We have also

$$\tfrac{1}{2}\,\mathrm{Tr}(\bar{\sigma}^\mu\bar{\sigma}_\nu{}^T)=g_\nu{}^\mu,$$

$$\sigma_\mu{}^T=C\bar{\sigma}_\mu C^{-1}\quad\text{or}\quad\sigma_\mu=C\bar{\sigma}_\mu{}^T C^{-1}.$$

For any 2 by 2 matrix M the relation $CM^TC^{-1}=M^{-1}\det M$ is an identity.

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Plasmons, Gauge Invariance, and Mass

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(Received 8 November 1962)

Schwinger has pointed out that the Yang-Mills vector boson implied by associating a generalized gauge transformation with a conservation law (of baryonic charge, for instance) does not necessarily have zero mass, if a certain criterion on the vacuum fluctuations of the generalized current is satisfied. We show that the theory of plasma oscillations is a simple nonrelativistic example exhibiting all of the features of Schwinger's idea. It is also shown that Schwinger's criterion that the vector field $m\neq 0$ implies that the matter spectrum before including the Yang-Mills interaction contains $m=0$, but that the example of superconductivity illustrates that the physical spectrum need not. Some comments on the relationship between these ideas and the zero-mass difficulty in theories with broken symmetries are given.

RECENTLY, Schwinger[1] has given an argument strongly suggesting that associating a gauge transformation with a local conservation law does not necessarily require the existence of a zero-mass vector boson. For instance, it had previously seemed impossible to describe the conservation of baryons in such a manner because of the absence of a zero-mass boson and of the accompanying long-range forces.[2] The problem of the mass of the bosons represents the major stumbling block in Sakurai's attempt to treat the dynamics of strongly interacting particles in terms of the Yang-Mills gauge fields which seem to be required to accompany the known conserved currents of baryon number and hypercharge.[3] (We use the term "Yang-Mills" in Sakurai's sense, to denote any generalized gauge field accompanying a local conservation law.)

The purpose of this article is to point out that the familiar plasmon theory of the free-electron gas exemplifies Schwinger's theory in a very straightforward manner. In the plasma, transverse electromagnetic waves do not propagate below the "plasma frequency," which is usually thought of as the frequency of long-wavelength longitudinal oscillation of the electron gas. At and above this frequency, three modes exist, in close analogy (except for problems of Galilean invariance implied by the inequivalent dispersion of longitudinal and transverse modes) with the massive vector boson mentioned by Schwinger. The plasma frequency is equivalent to the mass, while the finite density of electrons leading to divergent "vacuum" current fluctuations resembles the strong renormalized coupling of Schwinger's theory. In spite of the absence of low-frequency photons, gauge invariance and particle conservation are clearly satisfied in the plasma.

In fact, one can draw a direct parallel between the dielectric constant treatment of plasmon theory[4] and Schwinger's argument. Schwinger comments that the commutation relations for the gauge field A give us one sum rule for the vacuum fluctuations of A, while those for the matter field give a completely independent value for the fluctuations of matter current j. Since j is the source for A and the two are connected by field equations, the two sum rules are normally incompatible unless there is a contribution to the A rule from a free, homogeneous, weakly interacting, massless solution of the field equations. If, however, the source term is large enough, there can be no such contribution and the massless solutions cannot exist.

The usual theory of the plasmon does not treat the electromagnetic field quantum-mechanically or discuss vacuum fluctuations; yet there is a close relationship between the two arguments, and we, therefore, show that the quantum nature of the gauge field is irrelevant. Our argument is as follows:

The equation for the electromagnetic field is

$$p^2A_\mu=(k^2-\omega^2)A_\mu(\mathbf{k},\omega)=4\pi j_\mu(\mathbf{k},\omega).$$

[1] J. Schwinger, Phys. Rev. **125**, 397 (1962).
[2] T. D. Lee and C. N. Yang, Phys. Rev. **98**, 1501 (1955).
[3] J. J. Sakurai, Ann. Phys. (N. Y.) **11**, 1 (1961).
[4] P. Nozières and D. Pines, Phys. Rev. **109**, 741 (1958).

A given distribution of current j_μ will, therefore, lead to a response A_μ given by

$$A_\mu=\frac{4\pi}{k^2-\omega^2}j_\mu=\frac{4\pi}{p^2}j_\mu. \quad (1)$$

(1) is merely the statement that only the electromagnetic current can be a source of the field; it is required for general gauge invariance and charge conservation according to the usual arguments.

The dynamics of the matter system—of the plasma in that case, of the vacuum in the elementary particle problem—determine a second response function, the response of the current to a given electromagnetic or Yang-Mills field. Let us call this response function

$$j_\mu(\mathbf{k},\omega)=-K_{\mu\nu}(\mathbf{k},\omega)A_\nu(\mathbf{k},\omega). \quad (2)$$

By well-known arguments of gauge invariance, $K_{\mu\nu}$ must have a certain form: Schwinger points out that in the relativistic case it must be proportional to $p_\mu p_\nu-g_{\mu\nu}p^2$, and equivalent arguments give one the same form in superconductivity.[5] It will be convenient to consider, for simplicity, only the gauge

$$p_\mu A_\mu=0. \quad (3)$$

Then the response is diagonal: $K_{\mu\nu}=-g_{\mu\nu}K$. For a plasma with n carriers of charge e and mass M it is simply (in the limit $p\to 0$)

$$K=ne^2/M. \quad (4)$$

In an insulator the response is not relativistically invariant. If the insulator has magnetic polarizability α_m and electric α_e, the response equations may be written, in the gauge (3),

$$j_\mu=-\alpha_e p^2 A_\mu \quad \text{(longitudinal and time components)},$$
$$\mathbf{j}=-\alpha_m p^2\mathbf{A} \quad \text{(transverse components)}.$$

In a truly relativistic situation such as our normal picture of a vacuum, we expect

$$j_\mu=-\alpha p^2 A_\mu \quad (5)$$

to describe normal polarizable behavior.

Since we cannot turn off the interactions, we do not actually observe the responses (1), (2), or (5). If we insert a test particle, its field $A_\mu{}^e$ induces a current j_μ which in turn acts as the source for an internal field $A_\mu{}^i$:

$$j_\mu=-K(A_\mu{}^i+A_\mu{}^e),\quad A_\mu{}^i=+4\pi j_\mu/p^2,$$

or, the total field is modified to

$$A_\mu=[p^2/(p^2+4\pi K)]A_\mu{}^e. \quad (6)$$

The pole at which A propagates freely occurs at a mass (frequency)

$$m^2=-p^2=4\pi K, \quad (7)$$

which in a conductor is

$$m^2=\omega^2-k^2=\omega_p{}^2. \quad (8)$$

ω_p is the usual plasma frequency $(4\pi ne^2/M)^{1/2}$.

It is not necessary here to go in detail into the relationship between longitudinal and transverse behavior of the plasmon. In the limit $p\to 0$ both waves propagate according to (8). The longitudinal plasmon is generally thought of as entirely an attribute of the plasma, while the transverse ones are considered to result from modification of the propagation of real photons by the medium. This is reasonable in the classical case because the longitudinal plasmon disappears at a certain cutoff energy and has a different dispersion law; but in a Lorentz-covariant theory of the vacuum it would be indistinguishable from the third component of a massive vector boson of which the transverse photons are the two transverse components.

How, then, if we were confined to the plasma as we are to the vacuum and could only measure renormalized quantities, might we try to determine whether, before turning on the effects of electromagnetic interaction, A had been a massless gauge field and K had been finite? As far as we can see, this is not possible; it is, nonetheless, interesting to see what the criterion is in terms of the actual current response function to a perturbation in the Lagrangian

$$\delta L=j_\mu\delta A_\mu. \quad (9)$$

This will turn out to be identical to Schwinger's criterion. The original "bare" response function was K:

$$j_\mu=-K_{\mu\nu}\delta A_\mu.$$

Taking into account the interaction, however, we must correct for the induced fields and currents, and we get

$$j_\mu=-K'\delta A_\mu{}^e=-K[p^2/(p^2+4\pi K)]\delta A_\mu{}^e\to -(p^2/4\pi)\delta A_\mu{}^e,\quad p^2\to 0. \quad (10)$$

Thus, the new response to an applied perturbing field (9) is very like that of an ordinary polarizable medium. The only difference from an ordinary polarizable "vacuum" with bare response (5) is that in that case as $p\to 0$

$$K'\to -[\alpha/(1+4\pi\alpha)]p^2, \quad (11)$$

so that the coefficient of $p^2/4\pi$ is less than unity.

This criterion is precisely the same as Schwinger's criterion

$$\int B_1(m^2)dm^2=1,$$

where $B_1 m^2$ is the weight function for the current vacuum fluctuations. This can be shown by a simple dispersion argument. Schwinger expresses the unordered

[5] M. R. Schafroth, Helv. Phys. Acta 24, 645 (1951).

product expectation value of the current as

$$\langle j_\mu(x)j_\nu(x')\rangle=\int dm^2\, m^2B_1(m^2)\int\frac{dp}{(2\pi)^3}e^{ip(x-x')}\times\eta_+(p)\delta(p^2+m^2)(p_\mu p_\nu-g_{\mu\nu}p^2).$$

The Fourier transform of the corresponding retarded Green's function is our response function:

$$K'(p)=\int\frac{dm^2\, m^2B_1(m^2)}{p^2-m^2}[p_\mu p_\nu-g_{\mu\nu}p^2],$$

and

$$\lim_{p\to 0}K'(p)=(p_\mu p_\nu-g_{\mu\nu}p^2)\int dm^2\, B_1(m^2).$$

Thus, (aside from a factor 4π which Schwinger has not used in his field equation) his criterion is also that the polarizability α', here expressed in terms of a dispersion integral, have its maximum possible value, 1.

The polarizability of the vacuum is not generally considered to be observable[6] except in its p dependence (terms of order p^4 or higher in K). In fact, we can remove (11) entirely by the conventional renormalization of the field and charge

$$A_r=AZ^{-1/2},\quad e_r=eZ^{1/2},\quad j_r=jZ^{1/2}.$$

Z, here, can be shown to be precisely

$$Z=1-4\pi\alpha'=1-\int_0^\infty dm^2\, B_1(m^2).$$

Thus, the renormalization procedure is possible for any merely polarizable "vacuum," but not for the special case of the conducting "plasma" type of vacuum. In this case, no net true charge remains localized in the region of the dressed particle; all of the charge is carried "at infinity" corresponding to the fact, well known in the theory of metals, that all the charge carried by a quasi-particle in a plasma is actually on the surface. Nonetheless, conservation of particles, if not of bare charge, is strictly maintained. Note that the situation does not resemble the case of "infinite" charge renormalization because the infinity in the vacuum polarizability need only occur at $p^2=0$.

Either in the case of the polarizable vacuum or of the "conducting" one, no low-energy experiment, and even possibly no high-energy one, seems capable of directly testing the value of the vacuum polarizability prior to renormalization. Thus, we conclude that the plasmon is a physical example demonstrating Schwinger's contention that under some circumstances the Yang-Mills type of vector boson need not have zero mass. In addition, aside from the short range of forces and the finite mass, which we might interpret without resorting to Yang-Mills, it is not obvious how to characterize such a case mathematically in terms of observable, renormalized quantities.

We can, on the other hand, try to turn the problem around and see what other conclusions we can draw about possible Yang-Mills models of strong interactions from the solid-state analogs. What properties of the vacuum are needed for it to have the analog of a conducting response to the Yang-Mills field?

Certainly the fact that the polarizability of the "matter" system, without taking into account the interaction with the gauge field, is infinite need not bother us, since that is unobservable. In physical conductors we can see it, but only because we can get outside them and apply to them true electromagnetic fields, not only internal test charges.

More serious is the implication—obviously physically from the fact that α has a pole at $p^2=0$—that the "matter" spectrum, at least for the "undressed" matter system, must extend all the way to $m^2=0$. In the normal plasma even the final spectrum extends to zero frequency, the coupling rather than the spectrum being affected by the screening. Is this necessarily always the case? The answer is no, obviously, since the superconducting electron gas has no zero-mass excitations whatever. In that case, the fermion mass is finite because of the energy gap, while the boson which appears as a result of the theorem of Goldstone[7,8] and has zero unrenormalized mass is converted into a finite-mass plasmon by interaction with the appropriate gauge field, which is the electromagnetic field. The same is true of the charged Bose gas.

It is likely, then, considering the superconducting analog, that the way is now open for a degenerate-vacuum theory of the Nambu type[9] without any difficulties involving either zero-mass Yang-Mills gauge bosons or zero-mass Goldstone bosons. These two types of bosons seem capable of "canceling each other out" and leaving finite mass bosons only. It is not at all clear that the way for a Sakurai[3] theory is equally uncluttered. The only mechanism suggested by the present work (of course, we have not discussed non-Abelian gauge groups) for giving the gauge field mass is the degenerate vacuum type of theory, in which the original symmetry is not manifest in the observable domain. Therefore, it needs to be demonstrated that the necessary conservation laws can be maintained.

I should like to close with one final remark on the Goldstone theorem. This theorem was initially conjectured, one presumes, because of the solid-state analogs, via the work of Nambu[10] and of Anderson.[11] The theorem states, essentially, that if the Lagrangian

[6] We follow here, as elsewhere, the viewpoint of W. Thirring, *Principles of Quantum Electrodynamics* (Academic Press Inc., New York, 1958), Chap. 14.

[7] J. Goldstone, Nuovo Cimento **19**, 154 (1961).

[8] J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).

[9] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).

[10] Y. Nambu, Phys. Rev. **117**, 648 (1960).

[11] P. W. Anderson, Phys. Rev. **110**, 827 (1958).

possesses a continuous symmetry group under which the ground or vacuum state is not invariant, that state is, therefore, degenerate with other ground states. This implies a zero-mass boson. Thus, the solid crystal violates translational and rotational invariance, and possesses phonons; liquid helium violates (in a certain sense only, of course) gauge invariance, and possesses a longitudinal phonon; ferro-magnetism violates spin rotation symmetry, and possesses spin waves; superconductivity violates gauge invariance, and would have a zero-mass collective mode in the absence of long-range Coulomb forces.

It is noteworthy that in most of these cases, upon closer examination, the Goldstone bosons do indeed become tangled up with Yang-Mills gauge bosons and, thus, do not in any true sense really have zero mass. Superconductivity is a familiar example, but a similar phenomenon happens with phonons; when the phonon frequency is as low as the gravitational plasma frequency, $(4\pi G\rho)^{1/2}$ (wavelength$\sim 10^4$ km in normal matter) there is a phonon-graviton interaction: in that case, because of the peculiar sign of the gravitational interaction, leading to instability rather than finite mass.[12] Utiyama[13] and Feynman have pointed out that gravity is also a Yang-Mills field. It is an amusing observation that the three phonons plus two gravitons are just enough components to make up the appropriate tensor particle which would be required for a finite-mass graviton.

Spin waves also are known to interact strongly with magnetostatic forces at very long wavelengths,[14] for rather more obscure and less satisfactory reasons. We conclude, then, that the Goldstone zero-mass difficulty is not a serious one, because we can probably cancel it off against an equal Yang-Mills zero-mass problem. What is not clear yet, on the other hand, is whether it is possible to describe a truly strong conservation law such as that of baryons with a gauge group and a Yang-Mills field having finite mass.

I should like to thank Dr. John R. Klauder for valuable conversations and, particularly, for correcting some serious misapprehensions on my part, and Dr. John G. Taylor for calling my attention to Schwinger's work.

[12] J. H. Jeans, Phil. Trans. Roy. Soc. London **101**, 157 (1903).
[13] R. Utiyama, Phys. Rev. **101**, 1597 (1956); R. P. Feynman (unpublished).
[14] L. R. Walker, Phys. Rev. **105**, 390 (1957).

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Construction of Invariant Scattering Amplitudes for Arbitrary Spins and Analytic Continuation in Total Angular Momentum*

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(Received 9 November 1962)

From group-theoretical considerations, invariant scattering amplitudes for two-body reactions of particles with arbitrary spins and nonzero masses are constructed in various forms, including helicity amplitudes and amplitudes free of kinematical singularities. They are linear combinations of spin basis functions with scalar coefficients. In the process of construction the Pauli spin matrices are generalized for arbitrary spin. On the basis of a Mandelstam representation for the scalar coefficients, the unique analytic continuation of the amplitudes in total angular momentum is obtained. Possible kinematical singularities of the scalar amplitudes at the boundary of the physical region are discussed.

I. INTRODUCTION

The basic quantities of S-matrix theory are the Lorentz-invariant scattering matrix elements (S functions), which depend on the spins and types of incoming and outgoing particles and on the mass shell values of their four-momenta. From the S functions, invariant scattering amplitudes (M functions) that have simpler transformation properties and that are expected to be free of kinematical singularities can be defined.[1] A general procedure has been given to construct the invariant amplitudes in terms of the irreducible unitary representations of the inhomogeneous proper Lorentz group, based on a two-component spinor formalism.[2]

Although the invariant scalar amplitudes for which the Mandelstam representation is expected to be valid have been known for some time in the simpler cases such as those of the pion-nucleon[3] and nucleon-nucleon[4]

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[3] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).
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