Supplementary Material for "Quantum Annealed Criticality"

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Abstract

In this Supplementary Material we present details of the integration over the Gaussian strain fields for both the classical and quantum Larkin-Pikin approaches discussed in the main text.

I. OVERVIEW

The key idea of the Larkin-Pikin approach is that we integrate out the Gaussian strain degrees of freedom from the action to derive an effective action for the order parameter field so that

$$Z = \int \mathcal{D}[u] \int \mathcal{D}[\psi] e^{-S[\psi,u]} \qquad \longrightarrow \qquad Z = \int \mathcal{D}[\phi] \int \mathcal{D}[\psi] e^{-S_{eff}[\psi,\phi]}. \tag{1}$$

The crucial element in this procedure is a separation of the strain field into uniform and fluctuating components. When we integrate out the uniform component of the strain, it induces an infinite-range attractive interaction between the order parameter modes mediated by a *classical* field ϕ that is uniform in both space and time. The main effect of the integration of the fluctuating strain component is to renormalize the short-range interactions between the order parameter modes; however completion of the Gaussian integral also leads to an infinite range repulsive order parameter interaction. The overall infinite range interaction is attractive, but this subtlety needs to be checked carefully in both the classical and quantum cases, as is performed explicitly in this Supplementary Material; here we summarize its main results. The relevant quantum generalization of the effective action in (1) is

$$S_{eff}[\psi,\phi] = \int d^4x \left[\frac{\kappa}{2}\phi^2 + \frac{P^2}{2K} + \mathcal{L}[\psi, b^*] + \lambda \left(\phi + \frac{P}{K} \right) \psi^2[x] \right] \tag{2}$$

where

$$\mathcal{L}[\psi, b^*] = \frac{1}{2}(\partial_\mu \psi)^2 + \frac{a}{2}\psi^2 + \frac{b^*}{4!}\psi^4$$

is the ψ^4 Lagrangian, with a renormalized short-range interaction

$$b^* = b - \frac{12\lambda^2}{K + \frac{4}{3}\mu} \tag{3}$$

and an effective bulk modulus

$$\frac{1}{\kappa} = \frac{1}{K} - \frac{1}{K + \frac{4}{3}\mu}. \tag{4}$$

In the classical case

$$\int d^4x \longrightarrow \frac{1}{T} \int d^3x \tag{5}$$

and so we recover the classical effective action

$$S_{eff}[\psi,\phi] = \frac{1}{T} \int d^3x \left[\frac{\kappa}{2}\phi^2 + \frac{P^2}{2K} + \mathcal{L}[\psi, b^*] + \lambda(\phi + \frac{P}{K})\psi^2[\vec{x}] \right] \tag{6}$$

with definitions as above. We note that in the main text we have replaced the renormalized b^* by b and we have set $P = 0$ for presentational simplicity.

II. PRELIMINARIES

The partition function can be written as an integral over the order parameter and strain fields

$$Z = \int \mathcal{D}[\psi] \int \mathcal{D}[u] e^{-S[\psi,u]} \tag{7}$$

where ψ is the order parameter field and $u(x)$ is the local displacement of the lattice that determines the strain fields according to the relation

$$e_{ab}(x) = \frac{1}{2}\left(\frac{\partial u_a}{\partial x_b} + \frac{\partial u_b}{\partial x_a}\right). \tag{8}$$

Here the action is is determined by an integral over the Lagrangian L, $S = \int d^4 x L$. In the quantum case, $\int d^4 x \equiv \int d\tau \int d^3 x$ is a space-time integral over configurations that are periodic in imaginary time $\tau \in [0, \beta]$, where the β inverse temperature ($k_B = 1$). In the classical case the time-dependence disappears and the integral over τ is replaced by $1/T$ so that $S = \frac{1}{T}\int d^3 x L$.

The action divides up into three parts

$$S = S_A + S_I + S_B = \int d^4 x (L_A[u] + L_I[\psi, e] + L_B[\psi]), \tag{9}$$

where the contributions to the Lagrangian are: (i) a Gaussian term describing the elastic degrees of freedom in an isotropic system

$$L_A[u] = \frac{1}{2}\left[\rho \dot{u}_l^2 + \left(K - \frac{2}{3}\mu\right) e_{ll}^2 + 2\mu e_{ab}^2\right] - \sigma_{ab} e_{ab} \tag{10}$$

where σ_{ab} is the external stress and we have assumed a summation convention in which repeated indices are summed over, so that for instance, $e_{ll} \equiv \sum_{l=1,3} e_{ll}$ and $e_{ab}^2 \equiv e_{ab} e_{ab} = \sum_{a,b=1,3} e_{ab}^2$; (ii) an interaction term

$$L_I[\psi, e] = \lambda e_{ll} \psi^2 \tag{11}$$

describing the coupling between the volumetric strain $e_{ll} = \mathrm{Tr}[e]$ and the "energy density" ψ^2 of the order parameter ψ; (iii) the Lagrangian $L_B[\psi]$ of the order parameter that, in the simplest case, is a ψ^4 field theory

$$L_B[\psi, b] = \frac{1}{2}(\partial_\mu \psi)^2 + \frac{a}{2}\psi^2 + \frac{b}{4!}\psi^4, \tag{12}$$

where we have explicitly noted its dependence on the interaction strength b. At a finite temperature critical point, all time-derivative terms are dropped from these expressions.

Since the integral over the strain fields is Gaussian, the latter can be integrated out of the partition function leading to an effective action of the ψ fields $S_{eff}[\psi] = S_B[\psi] + \Delta S[\psi]$ where

$$e^{-\Delta S[\psi]} = \int \mathcal{D}[u] e^{-(S_A+S_I)}. \tag{13}$$

If we write the elastic action in the schematic, discretized form

$$S_A + S_I = \frac{1}{2}\sum_{i,j} u_i M_{ij} u_j + \lambda \sum_j u_j \psi_j^2 \tag{14}$$

then the effective action becomes simply

$$\Delta S = \frac{1}{2}\ln\det[M] - \frac{\lambda^2}{2}\sum_{i,j}\psi_i^2 M_{i,j}^{-1}\psi_j^2 \tag{15}$$

where the second term is recognized as an induced attractive interaction between the order-parameter fields. The subtlety in this procedure derives from the division of the strain field into two parts: a uniform contribution determined by boundary conditions and a fluctuating component in the bulk. For the classical case

$$e_{ab}(\vec{x}) = e_{ab} + \frac{1}{\sqrt{V}}\sum_{\vec{q}\neq 0}\frac{i}{2}\left(q_a u_b(\vec{q}) + q_b u_a(\vec{q})\right) e^{i\vec{q}\cdot\vec{x}} \tag{16}$$

where the $u_b(\vec{q})$ are the Fourier transform of the atomic displacements, while in the quantum problem

$$e_{ab}(\vec{x},\tau) = e_{ab} + \frac{1}{\sqrt{V\beta}}\sum_{i\nu_n}\sum_{\vec{q}\neq 0}\frac{i}{2}\left(q_a u_b(q) + q_b u_a(q)\right) e^{i(\vec{q}\cdot\vec{x}-\nu_n\tau)}, \tag{17}$$

where $\nu_n = 2\pi n T$ is the bosonic Matsubara frequency. Note that the exclusion of all terms where $\vec{q} = 0$ from the summation also excludes the special point where both $i\nu_n$ and $\vec{q}$ are zero. As we now demonstrate, the overall attractive interaction ($\propto -\psi_i^2 M_{ij}^{-1}\psi_j^2$) contains both short-range and infinite range components.

III. THE GAUSSIAN STRAIN INTEGRAL: CLASSICAL CASE

Our task is to calculate the Gaussian integral,

$$e^{-\Delta S[\psi]} = \int \mathcal{D}[e_{ab}, u_q] e^{-(S_A+S_I)} \tag{18}$$

where the classical action

$$S_A + S_I = \frac{1}{T}\int d^3x \left[\frac{1}{2}\left(K - \frac{2}{3}\mu\right) e_{ll}^2(\vec{x}) + \mu e_{ab}(\vec{x})^2 + (\lambda\psi^2(\vec{x}) + P) e_{ll}(\vec{x})\right], \tag{19}$$

where we have denoted $\sigma_{ab} = -P\delta_{ab}$ in terms of the pressure P. We begin by splitting the strain field into the $q = 0$ and finite q components,

$$e_{ab}(\vec{x}) = e_{ab} + \frac{1}{\sqrt{V}} \sum_{\vec{q} \neq 0} \frac{i}{2} \left(q_a u_b(\vec{q}) + q_b u_a(\vec{q}) \right) e^{i\vec{q}\cdot\vec{x}}. \tag{16}$$

This separation enables us to use periodic boundary conditions, putting the system onto a spatial torus with discrete momenta $\vec{q} = \frac{2\pi}{L}(l, m, n)$. After this transformation, the action divides up into two terms, $S = S[e_{ab}, \psi] + S[u, \psi]$. We shall define the integrals

$$\int de_{ab} e^{-S[e_{ab},\psi]} = e^{-S_1[\psi]},$$

and

$$\int \mathcal{D}[u] e^{-S[u,\psi]} = e^{-S_2[\psi]}. \tag{20}$$

The uniform part of the action is

$$\begin{aligned} S[e_{ab}, \psi] &= \frac{V}{T} \left[\frac{1}{2} \left(K - \frac{2}{3}\mu \right) e_{ll}^2 + \mu e_{ab}^2 \right] + \frac{V}{T} (\lambda \psi_{q=0}^2 + P) e_{ll} \\ &= \frac{1}{2} e_{ab} \mathcal{M}_{abcd} e_{cd} + v_{ab} e_{ab}, \end{aligned} \tag{21}$$

where $\psi_{\vec{q}}^2 = \frac{1}{V} \int d^3x \psi^2(\vec{x}) e^{i\vec{q}\cdot\vec{x}}$ is the Fourier transform of the fluctuations in "energy density" and

$$\mathcal{M}_{abcd} = K \overbrace{(\delta_{ab}\delta_{cd})}^{\mathcal{P}^L_{abcd}} + 2\mu \overbrace{\left(\delta_{ac}\delta_{bd} - \frac{1}{3}\delta_{ab}\delta_{cd} \right)}^{\mathcal{P}^T_{abcd}}, \tag{22}$$

$$v_{ab} = \frac{V}{T} (\lambda \psi_{q=0}^2 + P) \delta_{ab}. \tag{23}$$

The nonuniform part of the action is

$$S[u, \psi] = \frac{1}{T} \sum_{\vec{q} \neq 0} \left(\frac{1}{2} u_a^*(\vec{q}) M_{ab} u_b(\vec{q}) + \vec{a}(\vec{q}) \cdot \vec{u}(\vec{q}) \right) \tag{24}$$

where

$$\begin{aligned} M_{ab} &= \left[\left(K - \frac{2}{3}\mu \right) q_a q_b + \mu \left(q^2 \delta_{ab} + q_a q_b \right) \right], \\ \vec{a}_q &= \left(i\lambda\sqrt{V}\ \psi_{-q}^2 \right) \vec{q}. \end{aligned} \tag{25}$$

When we integrate over the uniform part of the strain field,

$$\frac{1}{2} e_{ab} \mathcal{M}_{abcd} e_{cd} + v_{ab} e_{ab} \to S_1[\psi] = -\frac{1}{2} v_{ab} \mathcal{M}^{-1}_{abcd} v_{cd} \tag{26}$$

Now the two terms P^L_{abcd} and P^T_{abcd} in $\mathcal{M}$ (22) are independent projection operators ($P^\Gamma_{abef}P^\Gamma_{efcd} = P^\Gamma_{abcd}$, $\Gamma \in L, T$), projecting the longitudinal and transverse components of the strain. The inverse of $\mathcal{M}$ is then given by

$$\mathcal{M}^{-1}_{abcd} = \frac{T}{V}\left[\frac{1}{K}(\delta_{ab}\delta_{cd}) + \frac{1}{2\mu}\left(\delta_{ac}\delta_{bd} - \frac{1}{3}\delta_{ab}\delta_{cd}\right)\right], \tag{27}$$

so the Gaussian integral over the uniform part of the strain field gives

$$S_1[\psi] = -\frac{1}{2}v_{ab}\mathcal{M}^{-1}_{abcd}v_{cd} = -\frac{V}{2T}\frac{1}{K}(\lambda\psi^2_{q=0} + P)^2. \tag{28}$$

Now the matrix entering the fluctuating part of the action $S[u,\psi]$, can be projected into the longitudinal and transverse components of the strain

$$M_{ab}(\vec{q}) = q^2\left[\left(K + \frac{4}{3}\mu\right)\hat{q}_a\hat{q}_b + \mu(\delta_{ab} - \hat{q}_a\hat{q}_b)\right] \tag{29}$$

where $\hat{q}_a = q_a/q$ are the direction cosines of $\vec{q}$. The inversion of this matrix is then

$$M^{-1}_{ab}(\vec{q}) = q^{-2}\left[\left(K + \frac{4}{3}\mu\right)^{-1}\hat{q}_a\hat{q}_b + \mu^{-1}(\delta_{ab} - \hat{q}_a\hat{q}_b)\right], \tag{30}$$

so the Gaussian integral over fluctuating part of the strain field leads to

$$\begin{aligned}\frac{1}{T}\sum_{\vec{q}\neq 0}\frac{1}{2}u^*_a(\vec{q})M_{ab}(\vec{q})u_b(\vec{q}) + \vec{a}(\vec{q})\cdot\vec{u}(\vec{q}) \rightarrow \\ S_2[\psi] &= -\frac{1}{2T}\sum_{\vec{q}\neq 0}a_a(-\vec{q})M^{-1}_{ab}(\vec{q})a_b(\vec{q}) \\ &= -\frac{V}{2T}\sum_{\vec{q}\neq 0}\psi^2_{-q}\psi^2_q\frac{\lambda^2}{K + \frac{4}{3}\mu}\end{aligned} \tag{31}$$

We can rewrite this as a sum over *all* $\vec{q}$, plus a remainder at $\vec{q} = 0$:

$$\begin{aligned}S_2[\psi] &= -\frac{V}{2T}\sum_{\vec{q}}\psi^2_{-q}\psi^2_q\frac{\lambda^2}{K + \frac{4}{3}\mu} + \frac{V}{2T}(\psi^2_{q=0})^2\frac{\lambda^2}{K + \frac{4}{3}\mu} \\ &= -\frac{1}{2T}\frac{\lambda^2}{K + \frac{4}{3}\mu}\int d^3x\psi^4(\vec{x}) + \frac{V}{2T}(\psi^2_{q=0})^2\frac{\lambda^2}{K + \frac{4}{3}\mu}.\end{aligned} \tag{32}$$

The first term is a local attraction while the second term, involving only the $\vec{q} = 0$ Fourier component, corresponds to to a repulsive long range interaction.

When we combine the results of the two integrals (28) and (32) we obtain

$$\Delta S[\psi] = -\frac{V}{2T}\frac{\lambda^2}{\kappa}(\psi^2_{q=0})^2 - \frac{1}{2T}\frac{\lambda^2}{K + \frac{4}{3}\mu}\int d^3x\psi^4(x) - \frac{V}{2T}\frac{1}{K}(2\lambda\psi^2_{q=0}P + P^2) \tag{33}$$

where

$$\frac{1}{\kappa} = \frac{1}{K} - \frac{1}{K + \frac{4}{3}\mu} \tag{34}$$

is the effective Bulk modulus.

The final step in the procedure, is to carry out a Hubbard Stratonovich transformation, factorizing the long-range attraction in terms of a stochastic uniform field ϕ,

$$-\frac{V}{2T}\frac{\lambda^2}{\kappa}(\psi_{q=0}^2)^2 \to \frac{1}{T}\int d^3x \left[\frac{\kappa}{2}\phi^2 + \lambda\phi\psi^2(x)\right]. \tag{35}$$

Combining (33) and (35) we obtain the following expression for

$$\Delta S[\psi,\phi] = \frac{1}{T}\int d^3x \left[\frac{\kappa}{2}\phi^2 + \frac{P^2}{2K} + \lambda\left(\phi + \frac{P}{K}\right)\psi^2(x) - \frac{\lambda^2}{2(K+\frac{4}{3}\mu)}\psi^4(x)\right]. \tag{36}$$

Finally, adding this term to the original order parameter action $S_B[\psi] = \frac{1}{T}\int d^3x L_B[\psi, b]$, our final partition function can be written

$$Z = \int d\phi \int \mathcal{D}[\psi] e^{-S_{eff}[\psi,\phi]} \tag{37}$$

where $S_{eff}[\psi,\phi] = S_B[\psi] + \Delta S[\psi,\phi]$ is given by

$$S_{eff}[\psi,\phi] = \frac{1}{T}\int d^3x \left[\frac{\kappa}{2}\phi^2 + \frac{P^2}{2K} + \mathcal{L}[\psi,b^*] + \lambda(\phi + \frac{P}{K})\psi^2[x]\right] \tag{38}$$

where

$$\mathcal{L}[\psi,b^*] = \frac{1}{2}(\partial_\mu\psi)^2 + \frac{a}{2}\psi^2 + \frac{b^*}{4!}\psi^4.$$

is the ψ^4 Lagrangian, with a renormalized short-range interaction

$$b^* = b - \frac{12\lambda^2}{K + \frac{4}{3}\mu}. \tag{39}$$

Note that in the main text we have dropped the "$*$" on b for presentational simplicity; there b refers to this renormalized interaction (39).

Thus the main effects of integrating out the strain field are a renormalization of the short-range interaction of the order parameter field and the development of an infinite-range interaction mediated by an effective strain field ϕ. If we differentiate the action with respect to the pressure, we obtain the volumetric strain

$$\frac{\delta S}{\delta P(\vec{x})} = e_{ll}(\vec{x}) = \frac{1}{K}\left(P + \lambda\psi^2(\vec{x})\right), \tag{40}$$

which, as a result of integrating out the strain fluctuations, now contains a contribution from the order parameter. Again in the main text we set $P = 0$ for presentational simplicity.

IV. THE GAUSSIAN STRAIN INTEGRAL: QUANTUM CASE

In the quantum case, the action in the Gaussian strain integral

$$e^{-\Delta S[\Psi]} = \int \mathcal{D}[e_{ab}, u_q] e^{-(S_A+S_I)} \tag{41}$$

now involves an integral over space time, with $S = \int d^4 x L \equiv \int_0^\beta d\tau \int d^3 x L$. We now restore the kinetic energy terms in Lagrangian (10) and (12), so that now the quantum action takes the form

$$S_A + S_I = \int d\tau d^3x \left[\frac{\rho}{2}\dot{u}_l^2 + \left(K - \frac{2}{3}\mu\right) e_{ll}^2(x) + \frac{1}{2} 2\mu e_{ab}(x)^2 + (\lambda\psi^2(x) + P) e_{ll}(x)\right]. \tag{42}$$

Again our task is to cast this into matrix form

$$S_A + S_I = \frac{1}{2}\sum_q u_i M_{ij} u_j + \lambda \sum_j u_j \psi_j^2 \rightarrow \frac{\lambda^2}{2}\sum_{i,j} \psi_i^2 M_{i,j}^{-1} \psi_j^2. \tag{43}$$

where now the summations run over the discrete wavevector and Matsubara frequencies $q \equiv (i\nu_n, \vec{q})$, where $\nu_n = \frac{2\pi}{\beta} n$, $\vec{q} = \frac{2\pi}{L}(j, l, k)$. As before, we must separate out the static, $\vec{q} = 0$ component of the strain tensor, writing

$$e_{ab}(x, \tau) = e_{ab} + \frac{1}{\sqrt{V\beta}} \sum_{i\nu_n} \sum_{\vec{q}\neq 0} \frac{i}{2} \left(q_a u_b(q) + q_b u_b(q)\right) e^{i(\vec{q}\cdot\vec{x} - \nu_n \tau)}, \tag{44}$$

Note that there is no time-dependence to the uniform part of the strain, since the boundary conditions are static. However the fluctuating component excludes $\vec{q} = 0$, but includes all Matsubara frequencies; with these caveats, the quantum integration of the strain fields closely follows that of the classical case.

Again the action divides up into two terms, $S = S[e_{ab}, \psi] + S[u, \psi]$, corresponding to the distinct unifom and finite $\vec{q}$ contributions to the strain. We shall again define the integrals

$$\int de_{ab} e^{-S[e_{ab},\psi]} = e^{-S_1[\psi]},$$

and

$$\int \mathcal{D}[u] e^{-S[u,\psi]} = e^{-S_2[\psi]}. \tag{45}$$

The uniform part of the action

$$S[e_{ab}, \psi] = \int d\tau \left[\frac{1}{2}\left(K - \frac{2}{3}\mu\right) e_{ll}^2 + \frac{1}{2} 2\mu e_{ab}^2\right] + \frac{V}{T}(\lambda\psi_{q=0}^2 + P) e_{ll}$$

$$= \frac{1}{2} e_{ab} \mathcal{M}_{abcd} e_{cd} + v_{ab} e_{ab}, \tag{46}$$

where

$$\begin{aligned} \mathcal{M}_{abcd} &= \left[K(\delta_{ab}\delta_{cd}) + 2\mu \left(\delta_{ac}\delta_{bd} - \frac{1}{3}\delta_{ab}\delta_{cd} \right) \right], \\ v_{ab} &= V\beta(\lambda\psi^2_{q=0} + P)\delta_{ab}, \end{aligned} \tag{47}$$

is unchanged, but now

$$\psi^2_q = \frac{1}{V\beta} \int d^4x \psi^2(x) e^{-i(\vec{q}\cdot\vec{x} - \nu_n \tau)} \tag{48}$$

is the space-time Fourier transform of the order parameter intensity. The non-uniform part is now

$$S[u,\psi] = \sum_{i\nu_n} \sum_{\vec{q}\neq 0} \left(\frac{1}{2} u^*_a(q) M_{ab} u_b(q) + \vec{a}(q)\cdot\vec{u}(q) \right), \tag{49}$$

where

$$\begin{aligned} M_{ab} &= \left[\rho\nu_n^2 \left(K - \frac{2}{3}\mu \right) q_a q_b + \mu\left(q^2\delta_{ab} + q_a q_b\right) \right], \\ \vec{a}_q &= \left(i\lambda\sqrt{V\beta}\ \psi^2_{-q} \right) \vec{q}. \end{aligned} \tag{50}$$

When we integrate over the uniform part of the strain field, we obtain

$$\begin{aligned} &\frac{1}{2} e_{ab} \mathcal{M}_{abcd} e_{cd} + v_{ab} e_{ab} \rightarrow \\ &\qquad S_1[\psi] = -\frac{1}{2} v_{ab} \mathcal{M}^{-1}_{abcd} v_{cd} \\ &\qquad\qquad = -\frac{V\beta}{2K}(\lambda\psi^2_{q=0} + P)^2, \end{aligned} \tag{51}$$

or

$$S_1[\psi] = -\frac{1}{2K} \int d^4x (\lambda\psi^2_{q=0} + P)^2. \tag{52}$$

For presentational simplicity, we will now set $P = 0$ since the role of pressure here follows that in the classical treatment already described.

The matrix entering the fluctuating part of the action can be projected into the longitudinal and transverse components

$$M_{ab} = \left[\left(\rho\nu_n^2 + (K + \frac{4}{3}\mu) \right) \hat{q}_a\hat{q}_b + \left(\rho\nu_n^2 + \mu\right)(\delta_{ab} - \hat{q}_a\hat{q}_b) \right], \tag{53}$$

where $\hat{q}_a = q_a/q$ is the unit vector. The inversion of this matrix is then

$$M^{-1}_{ab} = \left[\frac{1}{\rho(\nu_n^2 + c_L^2 q^2)} \hat{q}_a\hat{q}_b + \frac{1}{\rho(\nu_n^2 + c_T^2 q^2)} (\delta_{ab} - \hat{q}_a\hat{q}_b) \right], \tag{54}$$

where

$$c_L^2 = \frac{K + \frac{4}{3}\mu}{\rho}, \qquad c_T^2 = \frac{2\mu}{\rho} \tag{55}$$

are the longitudinal and transverse sound velocities. The two terms appearing in M^{-1} are recognized as the propagators for longitudinal and tranverse phonons.

When we integrate over the fluctuating component of the strain field, only the longitudinal phonons couple to the order parameter:

$$\frac{1}{2}\sum_{i\nu_n}\sum_{\vec{q}\neq 0} u_a^*(q) M_{ab}(q) u_b(q) + \vec{a}(q)\cdot\vec{u}(q) \rightarrow$$

$$\begin{aligned} S_2[\psi] &= -\frac{1}{2}\sum_{i\nu_n}\sum_{\vec{q}\neq 0} a_a(-q) M_{ab}^{-1}(q) a_b(q) \\ &= -\frac{V\beta\lambda^2}{2}\sum_{i\nu_n,\vec{q}\neq 0} \psi_{-q}^2\psi_q^2 \left(\frac{q^2}{\rho\nu_n^2 + (K+\frac{4}{3}\mu)q^2}\right) \end{aligned} \tag{56}$$

Now in this last term,

$$\left(\frac{q^2}{\rho\nu_n^2 + (K+\frac{4}{3}\mu)q^2}\right) \tag{57}$$

the $\vec{q}=0$ term vanishes for any finite ν_n, but in the case where $\nu_n = 0$, the limiting $\vec{q}\rightarrow 0$ form of this term is finite:

$$\left(\frac{q^2}{\rho\nu_n^2 + (K+\frac{4}{3}\mu)q^2}\right)\Bigg|_{\vec{q}\rightarrow 0} = \begin{cases} 0 & \nu_n \neq 0 \\ \frac{1}{K+\frac{4}{3}\mu} & \nu_n = 0. \end{cases} \tag{58}$$

We can thus replace

$$\sum_{i\nu_n,\vec{q}\neq 0} \psi_{-q}^2\psi_q^2 \left(\frac{q^2}{\rho\nu_n^2 + (K+\frac{4}{3}\mu)q^2}\right) \rightarrow \sum_{i\nu_n,\vec{q}} \psi_{-q}^2\psi_q^2 \left(\frac{q^2}{\rho\nu_n^2 + (K+\frac{4}{3}\mu)q^2}\right) - \frac{(\psi_{q=0}^2)^2}{K+\frac{4}{3}\mu}. \tag{59}$$

so that

$$S_2[\psi] = \frac{V\beta\lambda^2}{2(K+\frac{4}{3}\mu)}(\psi_{q=0}^2)^2 - \frac{V\beta\lambda^2}{2}\sum_{i\nu_n,\vec{q}} \psi_{-q}^2\psi_q^2 \left(\frac{q^2}{\rho\nu_n^2 + (K+\frac{4}{3}\mu)q^2}\right). \tag{60}$$

If we now combine S_1 and S_2, we obtain

$$S_1 + S_2 = \frac{V\beta\lambda^2}{2\kappa}(\psi_{q=0}^2)^2 - \frac{V\beta\lambda^2}{2}\sum_{q} \psi_{-q}^2\psi_q^2 \left(\frac{q^2}{\rho\nu_n^2 + (K+\frac{4}{3}\mu)q^2}\right). \tag{61}$$

where

$$\frac{1}{\kappa} = \frac{1}{K} - \frac{1}{K+\frac{4}{3}\mu} \tag{62}$$

is the effective Bulk modulus, as in the classical case.

Next we carry out a Hubbard-Stratonovich transformation, rewriting the the long-range attraction in terms of a stochastic static and uniform scalar field ϕ as follows

$$-\frac{V\beta}{2}\frac{\lambda^2}{\kappa}(\psi^2_{q=0})^2 \to \int d^4x \left[\frac{\kappa}{2}\phi^2 + \lambda\phi\psi^2(x)\right]. \tag{63}$$

The remaining interaction term can be divided up into two parts as follows

$$\sum_q \psi^2_{-q}\psi^2_q \left(\frac{q^2}{\rho\nu_n^2 + (K+\frac{4}{3}\mu)q^2}\right) = \frac{1}{K+\frac{4}{3}\mu}\sum_q \psi^2_{-q}\psi^2_q \left[1 - \left(\frac{\nu_n^2/c_L^2}{q^2+\nu_n^2/c_L^2}\right)\right]. \tag{64}$$

The first term inside the brackets is independent of momentum and frequency, leading to a finite local attraction term that will act to renormalize the b term in the Lagrangian $\mathcal{L}_{\psi,b}$ as in the classical case. The second term is a non-local and retarded interaction. Due to Lorentz invariance, simple power-counting shows that this term has the same scaling dimensionality as a local repulsive term, and thus it will not modify the critical behavior of the second-order phase transition.

If we transform back into into space-time co-ordinates, then we obtain

$$S_1 + S_2 = \int d^4x \left[\frac{\kappa}{2}\phi^2 + \lambda\phi\psi^2(x) - \frac{\lambda^2}{2(K+\frac{4}{3}\mu)}\psi^4(x)\right] + S_{NL} = S_{eff}[\psi,\phi] + S_{NL} \tag{65}$$

where

$$S_{NL} = \frac{\lambda^2}{2V\beta(K+\frac{4}{3}\mu)}\int d^4x d^4x' \partial_\tau(\psi^2)(x)V(x-x')\partial_\tau(\psi^2)(x') \tag{66}$$

and

$$V(x) = \int \frac{d^4q}{(2\pi)^4}\left(\frac{c_L^{-2}}{|\vec{q}|^2 + \nu_n^2/c_L^2}\right)e^{i(\vec{q}\cdot\vec{x} - i\nu_n\tau)} = \frac{1}{2\pi c_L^2}\frac{1}{(|\vec{x}|^2 + c_L^2\tau^2)} \tag{67}$$

is the non-local interaction mediated by the acoustic phonons. Then the final quantum partition function resulting from integrating out the strain fields in the quantum case can be written ($P \neq 0$)

$$Z = \int d\phi \int \mathcal{D}[\psi] e^{-S_{eff}[\psi,\phi] - S_{NL}} \tag{68}$$

where

$$S_{eff}[\psi,\phi] = \int d^4x \left[\frac{\kappa}{2}\phi^2 + \frac{P^2}{2K} + \mathcal{L}[\psi,b^*] + \lambda\left(\phi + \frac{P}{K}\right)\psi^2[x]\right] \tag{69}$$

and

$$\mathcal{L}[\psi,b^*] = \frac{1}{2}(\partial_\mu\psi)^2 + \frac{a}{2}\psi^2 + \frac{b^*}{4!}\psi^4.$$

is the ψ^4 Lagrangian, with a renormalized short-range interaction

$$b^* = b - \frac{12\lambda^2}{K+\frac{4}{3}\mu} \tag{70}$$

and an effective bulk modulus

$$\frac{1}{\kappa} = \frac{1}{K} - \frac{1}{K + \frac{4}{3}\mu}. \tag{71}$$

Thus, as in the classical case, the main effect of integrating out the strain field, is a renormalization of the short-range interaction of the order parameter field, and the development of an infinite range interaction, mediated by an effective strain field ϕ. The introduction of a nonlocal contribution with the same scaling dimensions as the ψ^4 term will not affect the properties of the fixed point, and thus it will not change the universality class of the fixed point, as in the classical Larkin-Pikin case. However we emphasize that in its quantum generalization the effective dimension of the theory is $d_{eff} = d + z$. Again we note that in the main text we have replaced the coefficient of the renormalized interaction b^* in (70) by b for presentational simplicity.