Quantum annealed criticality: A scaling description

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Experimentally there exist many materials with first-order phase transitions at finite temperature that display quantum criticality. Classically, a strain-energy density coupling is known to drive first-order transitions in compressible systems, and here we generalize this Larkin *et al.* [Zh. Eksp. Teor. Fiz. **56**, 1664 (1969) [Sov. Phys. JETP **29**, 891 (1969)]] mechanism to the quantum case. We show that if the $T = 0$ system lies above its upper critical dimension, the line of first-order transitions ends in a "quantum annealed critical point" where zero-point fluctuations restore the underlying criticality of the order parameter. The generalized Larkin-Pikin phase diagram is presented and experimental consequences are discussed.

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I. INTRODUCTION

The interplay of first-order phase transitions with quantum fluctuations is an active area [1–8] in the study of exotic quantum states near zero-temperature phase transitions [9–15]. In many metallic quantum ferromagnets, coupling of the local magnetization to the low-energy particle-hole excitations transforms a high-temperature continuous phase transition into a low-temperature discontinuous one, and the resulting classical tricritical points have been observed in many systems [1–8]. Experimentally, there also exist insulating materials that have classical first-order transitions that display quantum criticality [16–20], and here we provide a theoretical basis for this behavior. In a nutshell, we study a system with strain-energy density coupling [21] that has a line of first-order transitions at finite temperatures. We show that as the temperature is lowered, quantum fluctuations reduce the amplitudes of their thermal counterparts, weakening the first-order transition and "annealing" the system's elastic response, ultimately resulting in a $T = 0$ "quantum annealed" critical point. The generalized temperature (T) tuning parameter (g) field (h) phase diagram emerging from our study is presented in Fig. 1 where the field (h) is conjugate to the order parameter.

At a first-order transition the quartic mode-mode coupling of the effective action becomes negative. One mechanism for this phenomenon, studied by Larkin and Pikin [21] (LP), involves the interaction of strain with the fluctuating energy density of a critical order parameter. LP found that a diverging specific heat in the "clamped" (fixed volume) system leads to a first-order transition in the unclamped system at constant pressure. The Larkin-Pikin criterion [21] for a first-order phase transition is

$$\kappa \lesssim \frac{\Delta C_V}{T_c}\left(\frac{dT_c}{d\,\ln V}\right)^2, \tag{1}$$

where V is the volume, ΔC_V is the singular part of the specific-heat capacity in the clamped critical system, T_c is the transition temperature, and $\frac{dT_c}{d\,\ln V}$ is its volume strain derivative. The effective bulk modulus κ is defined by $\kappa^{-1} = K^{-1} - (K + \frac{4}{3}\mu)^{-1}$ where K and μ are the bare bulk and the shear moduli in the absence of coupling between the order parameter and strain; more physically, $\kappa \sim K\frac{c_L^2}{c_T^2}$ where c_L and c_T are the longitudinal and the transverse sound velocities [22]. An experimental setting for this behavior is provided by $BaTiO_3$ with a classical ferroelectric phase transition that is continuous when clamped and, due to electromechanical coupling, becomes first order when unclamped [23,24].

Low-temperature measurements on ferroelectric insulators provide a key motivation for our study [16–20]. At finite temperatures and ambient pressure these materials often display first-order transitions due to strong electromechanical coupling [24]; yet in many cases [16–20] their dielectric susceptibilities suggest the presence of pressure-induced quantum criticality associated with zero-temperature continuous transitions [16–20]. It is thus natural to explore whether a generalization of the Larkin-Pikin mechanism [21] with strain-energy density coupling can be developed to describe this phenomenon.

Here we generalize the Larkin-Pikin approach to include quantum zero-point fluctuations of the energy density, showing that it is the divergence of the energy fluctuations, both quantum and classical, that govern the LP mechanism. Quantum fluctuations introduce an additional time dimension into the partition function, which now sums over all space-time

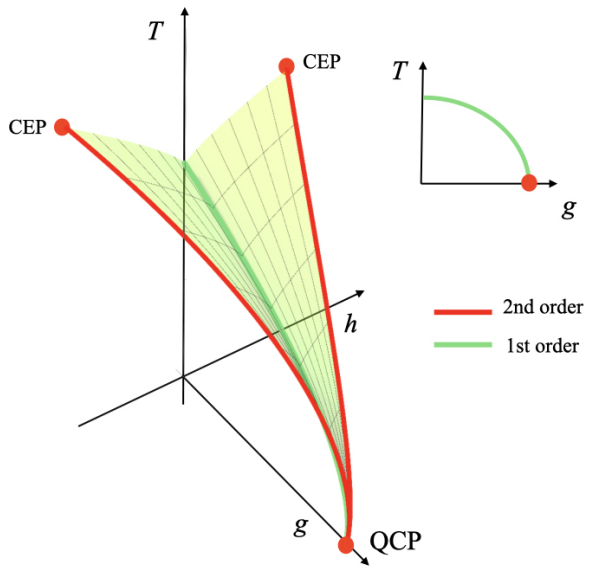

FIG. 1. Temperature (T) field (h) tuning parameter (g) phase diagram with a sheet of first-order transitions bounded by critical end points (CEP) terminating at a zero-temperature quantum critical point (QCP); here g tunes the quantum fluctuations and h is the field conjugate to the order parameter. Inset: temperature-tuning parameter "slice" indicating a line of classical phase transitions ending in a "quantum annealed critical point" where the underlying order-parameter criticality is restored by zero-point fluctuations.

configurations [25,26]. At a finite temperature T, the temporal extent of the quantum fluctuations is bounded by the Planck time $\tau_P = \frac{\hbar}{k_B T}$ with a corresponding quantum correlation length $\xi_Q \sim (\tau_P)^{1/z}$ where z is the the dynamical exponent. Therefore, for temperatures where ξ_Q is greater than the lattice spacing, the thermal correlation volume contains a quantum mechanical core on length scales and timescales determined by ξ_Q and τ_P. Due to their additional time dimension, quantum fluctuations are typically less singular than are their classical counterparts. As the temperature is lowered, the correlation volume of the zero-point fluctuations grows, reducing the amplitudes of the singular thermal fluctuations in the clamped system. The induced Larkin-Pikin first-order transition thus becomes progressively weaker with decreasing temperature, leading to a continuous "quantum annealed" transition at $T = 0$.

More specifically, Larkin and Pikin considered the coupling [21]

$$\mathcal{L}_I = \lambda e_{ll}(\vec{x})\psi^2(\vec{x}) \tag{2}$$

between the volumetric strain field e_{ll} and the squared amplitude ψ^2 of the critical order parameter. In a critical system, the singular fluctuations of the energy density are directly proportional to ψ^2; thus, (2) corresponds to a strain-energy coupling. Naively, (2) is expected to induce a short-range attractive order-parameter interaction. LP showed that (2) also leads to an anomalous long-range interaction between order-parameter fluctuations

$$S \longrightarrow S - \frac{\lambda^2}{2T\kappa}\left[\frac{1}{V}\int d^3x \int d^3x' \psi^2(\vec{x})\psi^2(\vec{x}')\right] \tag{3}$$

with

$$\frac{1}{\kappa} = \left(\frac{1}{K} - \frac{1}{K + \frac{4}{3}\mu}\right), \tag{4}$$

where μ is the shear modulus. This long-range interaction is finite if $\mu > 0$, i.e., if the medium is a solid. LP showed that this induced long-range interaction in (3) generates positive feedback to the tuning parameter, leading to a multivalued free-energy surface and a resulting first-order phase transition.

Here we expand the LP approach to include both quantum and classical fluctuations, summing over all possible space-time configurations in the action, to obtain a generalized LP criterion

$$\kappa \lesssim \left(\frac{dg_c}{d\ \ln V}\right)^2 \chi_{\psi^2}, \tag{5}$$

where

$$\chi_{\psi^2} = \int_0^\beta d\tau \int d^3x \langle \delta\psi^2(\vec{x})\delta\psi^2(0)\rangle \tag{6}$$

is the space-time average of the quantum and thermal "energy" fluctuations, $\beta = \frac{1}{k_B T}$, and g is the tuning parameter for the phase transition, with the convention that $g_c(T = 0) = 0$. At zero temperature, this expression extends the original LP criterion (1) to quantum phase transitions. At finite temperatures, the critical temperature and the critical coupling constant are related by $g_c(T_c) = uT_c^{1/\tilde{\Psi}}$, where $\tilde{\Psi} = \tilde{\nu}z$ is the shift exponent governing the finite-temperature transition with $\tilde{\nu}$ and z the quantum correlation length and the dynamical critical exponents, respectively [27]; therefore, $d\ \ln g_c = \frac{1}{\tilde{\Psi}} d\ \ln T_c$ and the LP criterion becomes

$$\kappa \lesssim \left(\frac{dT_c}{d\ \ln V}\right)^2 \overbrace{\left(\frac{g}{2T_c}\right)^2 \chi_{\psi^2}}^{\Delta C_V/T_c}, \tag{7}$$

where we have identified $\Delta C_v/T_c = (g/2T_c)^2 \chi_{\psi^2}$ with the specific-heat capacity. In this way, we see that the generalized Larkin-Pikin equation encompasses the original criterion (1) in addition to being applicable at low temperatures.

Recently, an adaptation of the Larkin-Pikin approach was proposed for pressure- (P-) tuned quantum magnets where it is often found that $\frac{dT_c}{dP} \to \infty$ as $T_c \to 0$. For a pressure-tuned transition $P - P_c = uT_c^{1/\tilde{\Psi}}$, so that $dT_c/dP \propto T_c^{1-1/\tilde{\Psi}}$ diverges as $T_c \to 0$ if $\tilde{\Psi} < 1$. It was then argued that the associated quantum phase transitions be first order [28–30]. However, such a diverging coupling of the critical order-parameter fluctuations and the lattice should lead to structural instabilities near the quantum phase transition that have not been observed [8,31]. Using Maxwell relations, we can write $\frac{dT_c}{dP} = \frac{\Delta V}{\Delta S}|_{T=T_c}$. Since $\Delta S \to 0$ as $T_c \to 0$, proponents of the previous argument assume that ΔV is finite in the same limit, indicating latent work at the quantum phase transition. However, the ratio $\frac{\Delta V}{\Delta S}$ can still diverge at a continuous quantum transition if the numerator and the denominator have different temperature dependencies as $T_c \to 0^+$. In fact, from our generalization of the Larkin-Pikin approach, we show that

$$\Delta V \propto -T^\eta, \quad \Delta S \propto -T^\eta (T^{\frac{1}{\tilde{\Psi}}-1}), \tag{8}$$

where $\eta = \frac{\alpha-\tilde{\alpha}}{\alpha\tilde{\Psi}}$ with α and $\tilde{\alpha}$ the classical and quantum critical exponents, respectively, governing the divergence of energy fluctuations. Generically, $\alpha > \tilde{\alpha}$ since thermal fluctuations are more singular than quantum fluctuations, so that $\eta > 0$. This means that

$$\lim_{T_c \to 0^+} \Delta V \to 0, \tag{9}$$

so there is no latent work at the quantum phase transition, confirming its continuous nature, despite the fact that when $\tilde{\Psi} < 1$,

$$\frac{dT_c}{dP} = \frac{\Delta V}{\Delta S}\bigg|_{T=T_c} \propto -T_c^{1-\frac{1}{\tilde{\Psi}}} \tag{10}$$

diverges as T_c goes to zero.

The structure of the paper is as follows. In Sec. II we present the original Larkin-Pikin approach [21], first constructing the classical LP action. Next, following LP, we parametrize the positive feedback contribution to the internal tuning parameter of the elastically coupled free energy using the uncoupled (clamped) free energy and scaling functions associated with classical criticality. The nonmonotonic relation between the internal tuning parameter and the physical temperature is coincident with a multivalued (unclamped) free energy, indicating the presence of a first-order transition in the elastically coupled system. We also derive the classical Larkin-Pikin criterion (1) as a macroscopic instability of the original (uncoupled) critical point with respect to the strain-energy density coupling [32]. This approach can be rewritten in terms of correlation functions, giving insight into the $T_c \to 0$ result.

The generalized Larkin-Pikin action is derived in Sec. III, where all possible space-time configurations are summed over so that quantum and thermal fluctuations are included. In Sec. IV a crossover scaling form for the clamped free energy that is applicable for both the classical and quantum critical points [27] is presented and used in the generalized Larkin-Pikin equations to study the system's behavior in the approach to $T_c \to 0$ along the clamped system's critical line. The Clausius-Clapeyron relations as $T_c \to 0$ are studied for the unclamped system, and it is shown that there is no latent work at $T_c \to 0$, confirming that the quantum transition is continuous. Next, a field conjugate to the order parameter is applied in Sec. V, and the critical end points are determined. Field behavior in the approach to the quantum critical point is also studied, and these results are summarized in the Larkin-Pikin phase diagram. Experimental consequences are presented in Sec. VI and we end (Sec. VII) with a summarizing discussion and open questions for future work. Derivations of the classical and quantum Larkin-Pikin actions and of various crossover scaling expressions are presented in five Appendices for interested readers.

II. CLASSICAL LARKIN-PIKIN APPROACH

The Larkin-Pikin (LP) mechanism [21] refers to a compressible system where the order parameter $\psi^2(\vec{x})$ is coupled to the volumetric strain in the simplest case of a scalar ψ and isotropic elasticity. The action $\mathscr{S}[\psi, u]$ for this compressible system then divides up into three contributions

$$\begin{aligned}\mathscr{S}[\psi, u] &= \mathscr{S}_L[\psi] + \mathscr{S}_E[u] + \mathscr{S}_I[\psi, e] \\ &= \frac{1}{T}\int d^3x(\mathcal{L}_L[\psi] + \mathcal{L}_E[u] + \mathcal{L}_I[\psi, e]).\end{aligned} \tag{11}$$

Here, in order to present the original classical LP problem in a way that is amenable to its quantum generalization considered later, we have used the notation $\mathcal{L}$, denoting the Lagrangian density which is also the Hamiltonian density in the classical case.

The Lagrangian density $\mathcal{L}_L[\psi]$ describes the physics of the order parameter in the clamped system that, in the simplest case, is a ψ^4 field theory

$$\mathcal{L}_L[\psi] = \frac{1}{2}(\partial_\mu\psi)^2 + \frac{a}{2}\psi^2 + \frac{b}{4!}\psi^4, \tag{12}$$

where $a = c(T - T_c)$ is the tuning parameter, and $b > 0$; the clamped system thus undergoes a continuous phase transition. The term

$$\mathcal{L}_E[u] = \frac{1}{2}\left[\left(K - \frac{2}{3}\mu\right)e_{ll}^2 + 2\mu e_{ab}^2\right] - \sigma_{ab}e_{ab} \tag{13}$$

describes the elastic degrees of freedom, where σ_{ab} is the external stress, $e_{ab}(\vec{x}) = \frac{1}{2}(\frac{\partial u_a}{\partial x_b} + \frac{\partial u_b}{\partial x_a})$ is the strain tensor, $u_a(\vec{x})$ is the local atomic displacement, and $e_{ll}(\vec{x}) = \mathrm{Tr}[e(\vec{x})]$ is the volumetric strain. Finally,

$$\mathcal{L}_I[\psi, e] = \lambda e_{ll}\psi^2 \tag{14}$$

describes the interaction between the volumetric strain e_{ll} and the squared amplitude ψ^2, the "energy density," of the order parameter, where λ is a coupling constant associated with the strain dependence of T_c. If we combine

$$\mathcal{L}_L + \mathcal{L}_I = \frac{1}{2}(\partial_\mu\psi)^2 + \frac{c}{2}(T - T_c[e_{ll}])\psi^2 + \frac{b}{4!}\psi^4, \tag{15}$$

where

$$T_c[e_{ll}] = T_c - (2\lambda/c)e_{ll} \tag{16}$$

is the strain dependent T_c, so that $(2\lambda/c) = -(\frac{dT_c}{d\,\mathrm{ln}V})$. For notational simplicity and convenience, we shall set $c = 1$ in the following development.

The key idea of the Larkin-Pikin approach is that we integrate out the Gaussian strain degrees of freedom from the action so that the partition function takes the form

$$Z = \int \mathcal{D}[\psi]\int \mathcal{D}[u]\, e^{-\mathscr{S}[\psi,u]} \longrightarrow Z = \int \mathcal{D}[\psi]e^{-S[\psi]}, \tag{17}$$

where the effective action S is a function of the order parameter ψ.

Although the elastic degrees of freedom are Gaussian, and can be exactly integrated out, this procedure must be done with some care because of the special role of boundary normal modes. In a solid of volume L^3, the normal modes can be separated into two components according to their wavelength λ: sound waves have wavelength $\lambda \ll L$ whereas boundary waves have wavelength comparable with the size of the system $\lambda \sim L$. The Larkin-Pikin effect is a kind of "*elastic anomaly,*" whereby the integration over boundary modes generates a nonlocal interaction between the order parameter

in the bulk action. This elastic anomaly destroys the locality of the original theory, yet, paradoxically, as a bulk term in the action it is independent of the detailed boundary conditions.

Larkin and Pikin chose periodic boundary conditions as the most convenient way to integrate out the boundary modes [21]. In a system with periodic boundary conditions, the strain field separates into a uniform ($\vec{q} = 0$) "boundary term" and a finite-momentum ($\vec{q} \neq 0$) contribution determined by fluctuating atomic displacements

$$e_{ab}(\vec{x}) = e_{ab} + \frac{1}{V}\sum_{\vec{q}\neq 0}\frac{i}{2}[q_a u_b(\vec{q}) + q_b u_a(\vec{q})]e^{i\vec{q}\cdot\vec{x}}, \tag{18}$$

where $u_a(\vec{q})$ is the Fourier transform of $u_a(\vec{x})$ with discrete momenta $\vec{q} = \frac{2\pi}{L}(l, m, n)$, with $\{l, m, n\}$ integers, $\{a, b\} \in \{1, 2, 3\}$ and volume $V = L^3$. Physically, we can understand this separation in (18) by noting that the strain only couples to the longitudinal modes; however, at $q = 0$ there is no distinction between transverse and longitudinal modes so this case must be treated separately from the finite-q situation. Formally, the solid forms a 3-torus, and the integral of the strain e_{ab} around the torus defines the number of line defects enclosed by the torus, a kind of flux, that is

$$\oint e_{ab}(\vec{x})dx_b = e_{ab}\oint dx_b = b_a, \tag{19}$$

where b_b is the Burgers vector of the enclosed defects. Thus, on a torus, the boundary modes of the strain have a topological character.

In order to integrate out the Gaussian strain degrees of freedom from (11) to derive an effective action for the order-parameter field in (17), we write the effective action

$$S[\psi] = S_L[\psi] + \Delta S[\psi], \tag{20}$$

where $S_L[\psi] = \frac{1}{T}\int d^3x\, \mathcal{L}_L[\psi]$ from (12) and

$$e^{-\Delta S[\psi]} = \int \mathcal{D}[e, u]e^{-(\mathscr{S}_E[u]+\mathscr{S}_I[\psi,e])}. \tag{21}$$

If we write the elastic action in a schematic, discretized form

$$\mathscr{S}_E[u] + \mathscr{S}_I[\psi, e] = \frac{1}{2}\sum_{i,j} u_i M_{ij} u_j + \lambda\sum_j u_j\psi_j^2, \tag{22}$$

then the effective action becomes simply

$$\Delta S[\psi] = \frac{1}{2}\ln\det[M] - \frac{\lambda^2}{2}\sum_{i,j}\psi_i^2 M_{i,j}^{-1}\psi_j^2, \tag{23}$$

where the second term is recognizable as an induced attractive interaction between the order-parameter fields. Because of subtleties associated with the separation (18) of the strain into uniform and finite-$\vec{q}$ components, integration of the elastic degrees of freedom in (21) leads to an overall attractive interaction ($\propto -\psi_i^2 M_{ij}^{-1}\psi_j^2$) with both short- and infinite-range components.

Integrating over the elastic degrees of freedom (18) in (21) we obtain the Larkin-Pikin action

$$S[\psi] = S_L[\psi, t, b^*] - \frac{\lambda^2}{2T}\left(\frac{1}{K} - \frac{1}{K+\frac{4}{3}\mu}\right)\left[\frac{1}{V}\int d^3x \int d^3x'\psi^2(\vec{x})\,\psi^2(\vec{x}')\right], \tag{24}$$

where

$$S_L(\psi) = \frac{1}{T}\int d^3x\left[\frac{1}{2}(\partial_\mu\psi)^2 + \frac{t}{2}\psi^2 + \frac{b^*}{4!}\psi^4\right] \tag{25}$$

with a renormalized local interaction

$$b^* = b - \frac{12\lambda^2}{K+\frac{4}{3}\mu}, \tag{26}$$

where we have made the replacement $a \to t$ where $t = (T - T_c)$ and $c = \frac{a}{T-T_c} = 1$.

The essence of the Larkin-Pikin effect is the appearance of a distance-independent interaction between the energy densities of the order-parameter field that appears in (24): it is this term that drives a nonperturbative first-order transition at arbitrarily small λ. Since the Larkin-Pikin argument is valid for arbitrarily small coupling λ, the perturbative $O(\lambda^2)$ renormalization of the short-range interaction in (26) becomes negligibly small in this limit and can be safely neglected. The prefactor of the long-range attractive interaction (24)

$$\frac{1}{\kappa} = \left(\frac{1}{K} - \frac{1}{K+\frac{4}{3}\mu}\right) \tag{27}$$

has two competing terms. The first is attractive ($\propto\frac{1}{K}$), resulting simply from integrating out the $q = 0$ part of the strain (18), governed by the bulk modulus K. The second results from the finite-q components of the strain (18), but in a rather subtle fashion. The finite-q elastic fluctuations, arising from longitudinal sound modes, are governed by the elastic modulus $K + \frac{4}{3}\mu$ and lead to the perturbative renormalization of b in (26). However, the finite-q modes explicitly exclude a contribution from $q = 0$. This "bosonic hole" in the longitudinal interactions gives rise to a residual long-range repulsion, resulting in the second term ($\propto -\frac{1}{K+\frac{4}{3}\mu}$). Remarkably, the overall prefactor of the long-range interaction term ($\frac{1}{\kappa}$) is only nonzero for finite shear modulus ($\mu \neq 0$), indicating that the Larkin-Pikin effect only occurs for solids and is absent for liquids. We also note that for the clamped system only the second contribution to (27) remains, leading to a repulsive

interaction; this is consistent with the continuous transition of the clamped system.

The distance-independent interaction in (24) can be written in terms of the volume average of the energy density

$$\Psi^2 \equiv \left[\frac{1}{V}\int d^3x\, \psi^2(\vec{x})\right] \tag{28}$$

so that

$$S[\psi] = S_L - \frac{\lambda^2 V}{2T\kappa}(\Psi^2)^2. \tag{29}$$

Since Ψ^2 is an intensive variable, its fluctuations about its thermal average $\langle\Psi^2\rangle$,

$$\delta\Psi^2 = \Psi^2 - \langle\Psi^2\rangle, \tag{30}$$

become vanishingly small in the thermodynamic limit, $\langle(\delta\Psi^2)\rangle \sim O(\frac{1}{V})$. Thus,

$$(\Psi^2)^2 = (\langle\Psi^2\rangle + \delta\Psi^2))^2 = 2\Psi^2\langle\Psi^2\rangle - \langle\Psi^2\rangle^2 + O(1/V^2), \tag{31}$$

so that we can reexpress (24) as a set of self-consistent equations

$$S[\psi] = \frac{1}{T}\int d^3x\left[\mathcal{L}_L(\psi,t) - \frac{\lambda^2}{\kappa}\langle\Psi^2\rangle\,\psi^2(\vec{x})\right] + \frac{\lambda^2 V}{2\kappa}\langle\Psi^2\rangle^2,$$
$$\langle\Psi^2\rangle = \frac{\int d\psi\,\Psi^2 e^{-S_L[\psi]}}{\int d\psi\, e^{-S_L[\psi]}}. \tag{32}$$

Equations (32) may be succinctly formulated by introducing an auxiliary "strain" variable

$$\phi = -\frac{\lambda\langle\Psi^2\rangle}{\kappa}. \tag{33}$$

Then, we may write

$$e^{-\frac{\tilde{F}(\phi)}{T}} = \int \mathcal{D}\psi\, e^{-S[\psi,\phi]}, \tag{34}$$

where

$$S[\psi,\phi] = \frac{1}{T}\int d^3x\left[\mathcal{L}_A(\psi,t) + \lambda\phi\psi^2 + \frac{\kappa}{2}\phi^2\right] \tag{35}$$

that can be reexpressed as

$$S[\psi,\phi] = \frac{1}{T}\int d^3x[\mathcal{L}_L(\psi,t+2\lambda\phi)] + \frac{\kappa V}{2T}\phi^2, \tag{36}$$

where we see that the auxiliary variable ϕ shifts the "mass" (e.g., tuning parameter) of the order parameter by $a \to x = a + 2\lambda\phi$. Self-consistency is then imposed as stationarity of the free energy with respect to ϕ:

$$\frac{\partial\tilde{F}[\phi]}{\partial\phi} = 0 \Longrightarrow [\lambda\langle\Psi^2\rangle + \kappa\phi]V = 0. \tag{37}$$

In the original Larkin-Pikin derivation [21], the action (35) was obtained by performing a Hubbard-Stratonovich transformation of the long-range interaction (29),

$$-\frac{\lambda^2 V}{2T}(\Psi^2)^2 \;\to\; \frac{\kappa V}{2T}\phi^2 \;+\; \lambda(\Psi^2)\phi, \tag{38}$$

followed by a saddle-point evaluation of the integral over ϕ. Larkin and Pikin observed that main effect of the elasticity in the unclamped system is to make a parametrized shift of the physical reduced temperature t to a parametrized variable X:

$$t \to X = t + \lambda\phi. \tag{39}$$

Although the phase transition of the unclamped system is continuous for parametrized parameter X, Larkin and Pikin [21] observed (see Sec. II A) that the original (physical) tuning parameter $t[X]$ becomes a nonmonotonic function of X, leading to a first-order phase transition at finite temperatures.

Subsequent authors pursued alternative approaches to the Larkin-Pikin criterion [32–34]. If, rather than integrating out the elasticity variable ϕ, one integrates out the order-parameter fluctuations, this results in a reduction $\Delta\kappa$ in the bulk modulus that is proportional to energy density fluctuations [32]. When the specific heat diverges, the bulk modulus is negative, there is a macroscopic instability, and the system undergoes a first-order transition. In the next two sections, we summarize each of these approaches to the classical Larkin-Pikin problem.

A. Review of the original Larkin-Pikin argument

The free energy of the clamped system is defined as

$$e^{-\frac{\mathcal{F}(t)}{T}} = \int \mathcal{D}[\psi]\, e^{-\mathscr{S}_L[\psi,t]}, \tag{40}$$

where $\mathscr{S}_L$ is defined in (11) and we have explicitly included its dependence on the tuning parameter t. In writing (40), we have glossed over issues of renormalization. In particular, self-energy corrections to the order-parameter propagators will shift the critical value of T_c in $t = T - T_c$ from its bare value T_c to a renormalized transition temperature T_c^*. All of these renormalization effects can be absorbed into redefinitions of the appropriate variables, in particular, from now on we will redefine $t = T - T_c^*$ and for convenience we will drop the asterisk so that T_c refers to the renormalized critical temperature.

From (34) and (35) we can write the free energy for our unclamped system as

$$\tilde{\mathcal{F}}[\phi,t] = \mathcal{F}[X] + \frac{\kappa V}{2}\phi^2, \tag{41}$$

where

$$X = t + 2\lambda\phi \tag{42}$$

indicates the shifting the of the tuning parameter due to the presence of energy fluctuations.

Now,

$$\frac{1}{V}\frac{\partial\mathcal{F}}{\partial X} = \frac{\langle\Psi^2\rangle}{2} \tag{43}$$

so that

$$\phi = -\frac{\lambda\langle\Psi^2\rangle}{\kappa} = -\frac{2\lambda}{V\kappa}\left(\frac{\partial\mathcal{F}}{\partial X}\right) \equiv -\frac{2\lambda}{V\kappa}\mathcal{F}'[X], \tag{44}$$

where we have defined $\mathcal{F}'[X] \equiv (\frac{\partial\mathcal{F}}{\partial X})$ for simplicity. Therefore,

$$\tilde{\mathcal{F}} = \mathcal{F}[X] + \frac{2\lambda^2}{V\kappa}(\mathcal{F}'[X])^2 \tag{45}$$

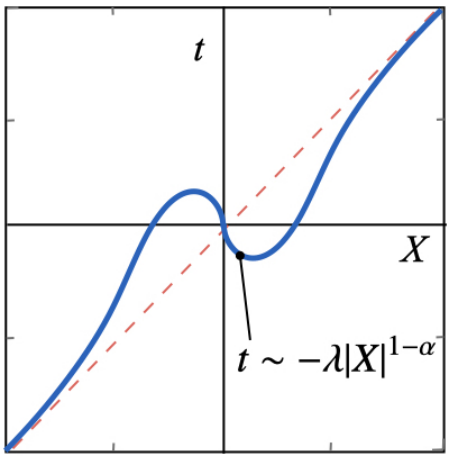

FIG. 2. Schematic of the nonmonotonic relationship between the reduced temperature (t) and the parametrized variable (X) shifted by energy fluctuations for the unclamped Larkin-Pikin problem.

and

$$X = t - \frac{4\lambda^2}{V\kappa}\mathcal{F}'[X]. \tag{46}$$

Let us define

$$\tilde{f} \equiv \frac{2\lambda}{V\kappa}\tilde{\mathcal{F}}, \quad f \equiv \frac{2\lambda}{V\kappa}\mathcal{F}. \tag{47}$$

Then, the two equations describing the unclamped system are

$$\tilde{f} = f[X] + \lambda(f'[X])^2 \tag{48}$$

and

$$t = X + 2\lambda f'[X] \tag{49}$$

which have to be solved self-consistently, where $\tilde{f}$ and f are the (renormalized) free energies of the unclamped and clamped systems, respectively.

To examine the consequences of these equations, we recall that that we can identify $a \propto \frac{T-T_c}{T_c} \equiv t$ with the reduced temperature t. In the clamped system, we assume a second-order transition so we can write

$$f \propto -|t|^{2-\alpha}, \tag{50}$$

where t is the reduced temperature and $\alpha > 0$ is the exponent associated with the specific heat. Since $a \propto t$, this leads to

$$f \propto -|X|^{2-\alpha} \quad \text{and} \quad f'[X] \propto -(2-\alpha)|X|^{1-\alpha}\text{sgn}(X) \tag{51}$$

and combining these results with (49) we obtain

$$t = X + 2\lambda f'[X] = X - 2\lambda(2-\alpha)|X|^{1-\alpha}\text{sgn}(X), \tag{52}$$

where we see that there is a *nonmonotonic* relationship between the physical temperature (t) and the parametrized variable (X), as shown in Fig. 2, that leads to an inevitable first-order transition.

In order to see more specifically how (52) translates into a discontinuous transition, let us consider, following the example of Larkin and Pikin [21], the specific case of $\alpha = \frac{1}{2}$. Then for t large, $f \propto |t|^{\frac{3}{2}}$. For $t = 0$ there are two solutions

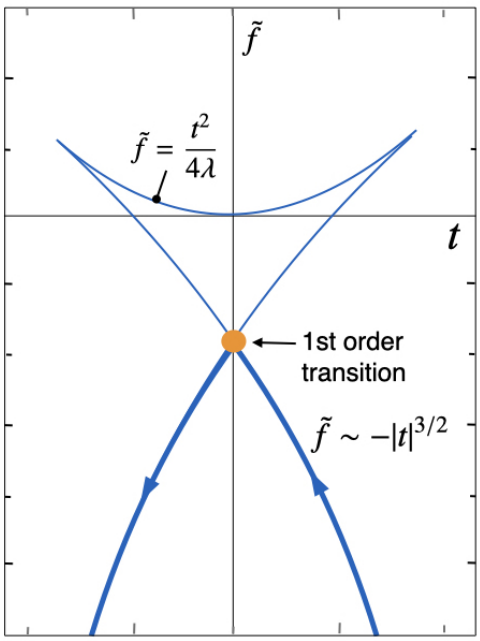

FIG. 3. Schematic of the free energy of the unclamped compressible system ($\tilde{f}$) vs reduced temperature (t) for $\alpha = \frac{1}{2}$; the first-order transition, due to the nonmonotonicity of t vs X, is marked here.

of (52): $X = 0$ and $4\lambda^2$ with $f = 0$ and $-\frac{16}{3}\lambda^3$, respectively. A plot of $\tilde{f}$ vs t is shown in Fig. 3, indicating the presence of a first-order transition.

The Larkin-Pikin criterion (1) emerges from

$$t = X + 2\lambda f'[X] = X + 2\frac{\lambda^2}{\kappa}\langle\Psi^2\rangle_X, \tag{53}$$

where again

$$\langle\Psi^2\rangle_X = \frac{\int \mathcal{D}\psi\,\Psi^2 e^{-S(X,\psi)}}{\int \mathcal{D}\psi\, e^{-S(X,\psi)}} \tag{54}$$

is the energy density computed with the shifted reduced temperature X. This expression describes the relationship between the physical temperature $t = (T - T_c)/T_c$ and the parametrized variable X. Now, the derivative dt/dX is given by

$$\frac{dt}{dX} = 1 - \frac{\lambda^2 V}{\kappa}\langle(\delta\Psi^2)^2\rangle. \tag{55}$$

We can identify the fluctuations in the right-hand side of this equation with the specific-heat capacity

$$C_V = \frac{\langle(\delta\Psi^2)^2\rangle}{T_c^2} \tag{56}$$

so that

$$\frac{dt}{dX} = 1 - \frac{\lambda^2 T_c^2}{\kappa}C_V(X). \tag{57}$$

Thus, if the specific-heat capacity diverges at the classical critical point of the clamped system $C_V(X) \to \infty$, dt/dX will change sign as $X \to 0$. Then, $t[X]$ becomes a nonmonotonic function of the internal temperature X that inevitably leads

to a first-order phase transition. $\frac{dt}{dX} = 0$ in (57) is the LP criterion (1).

B. Larkin-Pikin criterion as a macroscopic instability

An alternative approach to the Larkin-Pikin criterion is to probe the macroscopic stability of the original critical point with respect to the strain-energy density coupling [32]. From (34) and (35) we know that the partition function of the unclamped system can be written as an integral over the order-parameter fluctuations

$$Z[\phi] = e^{-\tilde{F}[\phi]/T} = \int \mathcal{D}[\psi]\, e^{-S[\psi,\phi]}. \tag{58}$$

The renormalized bulk modulus

$$\tilde{\kappa} \equiv \kappa - \Delta\kappa \tag{59}$$

at the transition is then obtained by taking the second derivative of (34),

$$\tilde{\kappa} = \frac{1}{V}\frac{\partial^2 \tilde{F}}{\partial \phi^2} = \kappa - \frac{\lambda^2}{T_c}\int d^3x \langle \delta\psi^2(\vec{x})\delta\psi^2(0)\rangle, \tag{60}$$

where $\delta\psi^2(\vec{x}) = \psi^2(\vec{x}) - \langle\psi^2\rangle$. We recall the tuning parameter $a = c(T - T_c)$ where we have set $c = 1$, so that the singular component of the specific-heat coefficient is also proportional to the energy fluctuations

$$\frac{\Delta C_V}{T_c} = -\frac{\partial^2 F}{\partial T^2}\bigg|_{\rm Sing} = \frac{1}{4T_c}\int d^3x \langle \delta\psi^2(\vec{x})\delta\psi^2(0)\rangle \tag{61}$$

which allows us to relate the shift in the bulk modulus to the singular part of the specific heat

$$\Delta\kappa = (2\lambda)^2 \frac{\Delta C_V}{T_c} = \left(\frac{dT_c}{d\ \ln V}\right)^2 \frac{\Delta C_V}{T_c}, \tag{62}$$

where we have used (78) to identify $2\lambda = -dT_c/d\ \ln V$. The condition for a macroscopic instability, and hence a first-order transition, is when the renormalized bulk modulus is negative

$$\kappa - \Delta\kappa < 0 \Rightarrow \quad \kappa \ < \ \frac{\Delta C_V}{T_c}\left(\frac{dT_c}{d\ \ln V}\right)^2 \tag{63}$$

and we see that we have recovered the Larkin-Pikin criterion (1).

The renormalization of the bulk modulus (59) that results can also be obtained diagrammatically (Fig. 4). In this approach the bare order-parameter interaction b now acquires a contribution from the coupling to the strain [Fig. 4(a)]. In the Feynman diagrams $1/\kappa$ is the bare "propagator" for the auxiliary strain variable ϕ. We can use a Dyson equation for this strain propagator [Fig. 4(b)] to determine $\tilde{\kappa}$. Specifically, we write

$$\left(\frac{1}{\tilde{\kappa}}\right) = \left(\frac{1}{\kappa}\right) + \left(\frac{1}{\kappa}\right)\lambda^2 \langle \psi^2(\vec{q})\psi^2(-\vec{q})\rangle|_{\vec{q}=0}\left(\frac{1}{\tilde{\kappa}}\right) \tag{64}$$

that results in

$$\tilde{\kappa} = \kappa - \Delta\kappa = \kappa - \lambda^2 \chi_{\psi^2}, \tag{65}$$

where χ_{ψ^2} is the static susceptibility for ψ^2,

$$\chi_{\psi^2} = \frac{1}{T_c}\int d^3x \langle \delta\psi^2(\vec{x})\delta\psi^2(0)\rangle, \tag{66}$$

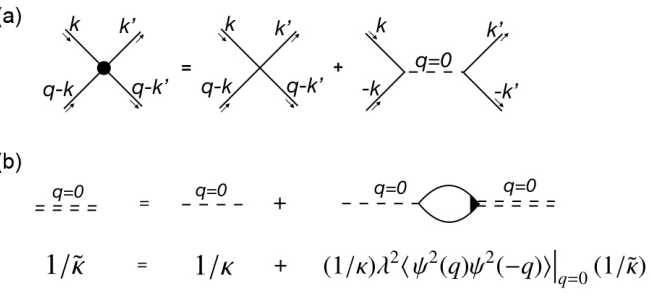

FIG. 4. Diagrammatic approach to the generalized Larkin-Pikin criterion. (a) Bare interaction is a sum of a local and a nonlocal contribution mediated by fluctuations in the strain. (b) Feynman diagram showing renormalization of the strain propagator by coupling to energy fluctuations.

recovering (60). $\Delta\kappa$ is thus a self-energy correction to the strain propagator.

This discussion enables us to obtain a heuristic understanding of how the Larkin-Pikin approach can be generalized to include quantum fluctuations of the order parameter which now occur in both both space and (imaginary) time. The prefactor $1/T_c$ in (66) is now replaced by an integral over time so that

$$\chi_{\psi^2} \longrightarrow \int_0^\beta d\tau \int d^dx \langle \delta\psi^2(\vec{x},\tau)\delta\psi^2(0)\rangle, \tag{67}$$

where we have also generalized the expression to d spatial dimensions. This quantity is represented by the *same* Feynman diagrams, where momentum variables now become four-momenta $q = (\vec{q}, \nu_n)$. If we make the Gaussian approximation $\langle \delta\psi^2(\vec{x})\delta\psi^2(0)\rangle \approx (\langle \delta\psi(\vec{x})\delta\psi(0)\rangle)^2$, then the zero-temperature limit of χ_{ψ^2} is

$$\begin{aligned}\lim_{T\to 0}\chi_{\psi^2} &\approx \int d\tau\, d^dx (\langle \delta\psi(\vec{x})\delta\psi(0)\rangle)^2 \\ &= \int \frac{d\nu}{2\pi}\frac{d^dq}{(2\pi)^d}(\chi_\psi(\vec{q},\nu))^2,\end{aligned} \tag{68}$$

where in the second line we have Fourier transformed into momentum space, and $\chi_\psi(\vec{q},\nu) = \langle \delta\psi(-q)\delta\psi(q)\rangle$, the order-parameter susceptibility is the space-time Fourier transform of the correlator $\langle\psi(\vec{x})\psi(0)\rangle$. It follows that

$$\lim_{T\to 0}\Delta\kappa \propto \int dq\, d\nu\, q^{d-1}\,[\chi_\psi(\vec{q}, i\nu)]^2. \tag{69}$$

To examine how this quantity behaves in the approach to the quantum critical point of the clamped system, we can use dimensional power counting. Since $[\chi] = [\frac{1}{q^2}]$ and $[\nu] = [q^z]$,

$$\lim_{T\to 0}[\Delta\kappa] = \frac{[q^{d+z}]}{[q^4]} \sim \xi_Q^{4-(d+z)},$$

where we have replaced $[q^{-1}] = [\xi_Q]$, the quantum correlation length. As the quantum critical point of the clamped system is approached, $\xi_Q \to \infty$, so that the quantum corrections to κ are nonsingular for $d + z > 4$.

III. GENERALIZATION OF THE LARKIN-PIKIN APPROACH TO INCLUDE QUANTUM FLUCTUATIONS

Motivated by these heuristic arguments, we now generalize the Larkin-Pikin approach to include both quantum and thermal fluctuations. In order to probe how the fluctuation-driven first-order transition predicted by Larkin and Pikin [21] evolves as the temperature is lowered to absolute zero, we need to understand the crossover between the continuous quantum and classical phase transitions of the clamped system. From a scaling perspective, temperature is a relevant perturbation that drives the system from a quantum to a classical critical point. The action of temperature on a quantum phase transition is to introduce a boundary condition in time, so that temperature plays the role of a finite-size correction at a quantum critical point. In contrast to their static classical counterparts, quantum zero-point fluctuations are intrinsically dynamical. At a finite temperature T, the criticality of quantum fluctuations is cut off by the Planck time $\tau_P = \frac{\hbar}{k_B T}$, with a corresponding quantum correlation length $\xi_Q \sim \tau_P^{1/z}$ where z is the dynamical exponent; thus, the static classical correlation volume contains a quantum mechanical core on length scales and timescales governed by the Planck time. At low temperatures, ξ_Q provides the essential short-distance cutoff to the static classical fluctuations of the order parameter. In pictorial terms, we can visualize the fluctuations as being "annealed" at short distances.

When the temperature is raised from absolute zero, there comes a point where the finite correlation time becomes of order the Planck time, and as the temperature is raised further, the temporal correlation length becomes "stuck" at the Planck time. The temperature when this occurs determines the quantum classical crossover. Beyond this point, correlations continue to grow but only in the spatial direction; the dynamical aspect of the fluctuations is lost and the statistical mechanics is governed purely by a sum over spatial configurations, namely, the statistical mechanics has become classical.

More specifically near the quantum critical point at $T = 0$, the zero-point fluctuations are governed by a finite correlation length $\xi_Q \sim [g - g_c(0)]^{-\tilde{\nu}}$, where g is the parameter that tunes the transition and $g = g_c(0)$ is the location of the quantum critical point. If we combine our expressions for the quantum correlation length in the ordered phase close to the line of phase transitions, we find

$$(g - g_c)^{-\tilde{\nu}} \sim \left(\frac{\hbar}{k_B T_c}\right)^{\frac{1}{z}} \tag{70}$$

which leads to

$$T_c \sim (g - g_c)^{\tilde{\nu} z} \equiv (g - g_c)^{\tilde{\Psi}}, \tag{71}$$

where $\tilde{\Psi}$ is called the shift exponent, and we see that $\Psi = \tilde{\nu} z$ if the effective dimension of the quantum system is below its upper critical dimension where scaling is applicable. Here, we keep with convention using this notation, hoping that there will be no confusion with the the space-time volume average of the energy density.

Larkin and Pikin [21] showed that the feedback effect of the energy fluctuations could be reformulated in terms of the critical temperature dependence of the free energy of the decoupled system near the phase transition, allowing an analysis purely in terms of the universal critical behavior of the decoupled system. By generalizing this parametric approach to include the effect of quantum fluctuations, we are able to analyze the evolution of the Larkin-Pikin system from finite to zero temperature, showing that if the energy fluctuations are not divergent at $T = 0$, the finite-temperature first-order phase transition progressively weakens as temperature is reduced, becoming continuous at zero temperature.

A. Generalized Larkin-Pikin action

The quantum mechanical action now picks up an additional integral over time

$$\mathcal{S} = \int d^4x\, \mathcal{L} \equiv \int_0^\beta d\tau \int d^3x\, \mathcal{L}, \tag{72}$$

where $\beta = \frac{1}{T}$ and we recover the classical result for large T. The space-time generalizations of Eqs. (12)–(14)

$$\begin{aligned}\mathcal{S}[\psi, u] &= \mathcal{S}_L[\psi] + \mathcal{S}_E[u] + \mathcal{S}_I[\psi, e] \\ &= \int d\tau\, d^3x (\mathcal{L}_A[\psi] + \mathcal{L}_E[u] + \mathcal{L}_I[\psi, e])\end{aligned} \tag{73}$$

now contain kinetic energy terms, so that

$$\mathcal{L}_L[\psi, b] = \frac{1}{2}(\partial_\mu \psi)^2 + \frac{a}{2}\psi^2 + \frac{b}{4!}\psi^4, \tag{74}$$

where $(\partial_\mu \psi) \equiv (\dot{\psi})^2 + (\nabla \psi)^2$ and we now identify $a = c(g - g_c^0)$, where g_c^0 is the bare value of the critical coupling constant. The elastic degrees of freedom are now described by

$$\mathcal{L}_E[u] = \frac{1}{2}\left[\rho \dot{u}_l^2 + \left(K - \frac{2}{3}\mu\right)e_{ll}^2 + 2\mu e_{ab}^2\right] - \sigma_{ab} e_{ab}, \tag{75}$$

and strain-energy density interaction

$$\mathcal{L}_I[\psi, e] = \lambda e_{ll} \psi^2 \tag{76}$$

is unchanged. If we combine

$$\mathcal{L}_L + \mathcal{L}_I = \frac{1}{2}(\partial_\mu \psi)^2 + \frac{c}{2}(g - g_c[e_{ll}])\psi^2 + \frac{b}{4!}\psi^4, \tag{77}$$

where

$$g_c[e_{ll}] = g_c^0 - (2\lambda/c) e_{ll} \tag{78}$$

is the strain dependent g_c, so that $(2\lambda/c) = -(\frac{dg_c}{d\ln V})$. For notational simplicity and convenience, we shall set $c = 1$ in the following development.

Following the argument of Larkin and Pikin in the classical case, we choose periodic boundary conditions as the most convenient way to integrate out the elastic degrees of freedom for the analogous quantum problem. It is then natural to generalize the classical expression for the strain field (18) to the quantum case summing over all space-time configurations as

$$\begin{aligned}e_{ab}(\vec{x}, \tau) = e_{ab}(\tau) + \frac{1}{\beta V} \sum_{i\nu_n} \sum_{\vec{q} \neq 0} \\ \times \frac{i}{2}[q_a u_b(q) + q_b u_a(q)] e^{i(\vec{q}\cdot\vec{x} - \nu_n \tau)},\end{aligned} \tag{79}$$

where $q_\alpha \equiv (\vec{q}, i\nu_n)$, $u_b(q) \equiv u_b(\vec{q}, i\nu_n)$, and $\nu_n = 2\pi nT$ is the Matsubara bosonic frequency. *A priori*, the uniform strain tensor $e_{ab}(\tau)$ involves configurations that are time dependent. However, we recall that the integral of the strain field around the toroidal solid

$$\oint e_{ab}(\vec{x}, \tau) dx_b = e_{ab}(\tau) \oint dx_b = b_a(\tau) \tag{80}$$

measures $b_a(\tau)$, the Burgers vector of the defects enclosed by the torus. If we restrict ourselves to smooth Gaussian deformations of the solid, then the Burgers vector is a topological invariant, like the conserved winding number of superconductor. Changes in the Burgers vector are akin to flux creep in a superconductor, and they involve the passage of dislocations across the entire solid. Space-time configurations with such moving defects will be associated with large actions, making their contributions to the path integral exponentially small in the thermodynamic limit. Therefore, the strain field for the quantum Larkin-Pikin problem can be written

$$e_{ab}(\vec{x}, \tau) = e_{ab} + \frac{1}{\beta V} \sum_{i\nu_n} \sum_{\vec{q}\neq 0} \times \frac{i}{2}[q_a u_b(q) + q_b u_a(q)] e^{i(\vec{q}\cdot\vec{x} - \nu_n \tau)}. \tag{81}$$

As in the classical case, our next step is to integrate out the Gaussian elastic degrees of freedom from the action

$$Z = \int \mathcal{D}[\psi] \int \mathcal{D}[u]\, e^{-\mathscr{S}[\psi, u]} \longrightarrow Z = \int \mathcal{D}[\psi] e^{-S[\psi]}, \tag{82}$$

where the actions now involve integrals over space-time. We write the effective action

$$S[\psi] = S_L[\psi] + \Delta S[\psi], \tag{83}$$

where $S_L[\psi] = \int d^4x\, \mathcal{L}_L[\psi]$ [(74)] and

$$e^{-\Delta S[\psi]} = \int \mathcal{D}[e, u] e^{-(\mathscr{S}_e[u] + \mathscr{S}_I[\psi, e])}. \tag{84}$$

Again, our task is to cast this action into matrix form

$$\mathscr{S}_E + \mathscr{S}_I = \frac{1}{2} \sum_q u_i M_{ij} u_j + \lambda \sum_j u_j \psi_j^2 \to \frac{\lambda^2}{2} \sum_{i,j} \psi_i^2 M^{-1}_{i,j} \psi_j^2 \tag{85}$$

and to determine the nature of the induced order-parameter interaction; now the summations run over the discrete wave vector and Matsubara frequencies $q \equiv (i\nu_n, \vec{q})$, where $\nu_n = \frac{2\pi}{\beta} n$, $\vec{q} = \frac{2\pi}{L}(j, l, k)$. Because of the form of the strain tensor (81), the action in (84) separates in two terms, corresponding to the $q = 0$ and the finite $(\vec{q}, i\nu_n)$ contributions. Integration over the elastic degrees of freedom in (81) now results in order-parameter interactions local and nonlocal both in space and time.

Integrating over the elastic degrees of freedom (81) in (84) (see Appendix B for details), we obtain the effective action

$$S[\psi] = S_L^*[\psi] - \frac{\lambda^2}{2}\left(\frac{1}{K} - \frac{1}{K + \frac{4}{3}\mu}\right) \times \left[\frac{1}{\beta V} \int d^4x \int d^4x' \psi^2(\vec{x})\, \psi^2(\vec{x}')\right]. \tag{86}$$

Here,

$$S_L^*(\psi) = \int d^4x \left[\frac{1}{2}(\partial_\mu \psi)^2 + \frac{\tilde{g}}{2}\psi^2 + \frac{b^*}{4}\psi^4 + \frac{1}{2}\psi^2(\vec{x}) V_{dyn}(x - x') \psi^2(\vec{x}')\right], \tag{87}$$

where $\tilde{g} = (g - g_c^0)$ and

$$b^* = b - \frac{12\lambda^2}{K + \frac{4}{3}\mu} \tag{88}$$

is identical to that in the classical case (26). The Fourier transform of $V_{dyn}(x - x')$ is

$$V_{dyn}(q) = \frac{\lambda^2}{K + \frac{4}{3}\mu}\left(\frac{\nu_n^2/c_L^2}{\vec{q}^2 + \nu_n^2/c_L^2}\right), \tag{89}$$

a dynamical order-parameter interaction where $q \equiv (\nu_n, \vec{q})$ is the wave vector in space-time. The effective action in the quantum Larkin-Pikin problem is thus the sum of a $(d + z)$-dimensional generalization of the classical effective LP action and a dynamical interaction induced by quantum fluctuations. Although $V_{dyn}(q)$ in (89) has q dependence, it is still local from a scaling perspective since $V_{dyn}(q)$ is finite and nonsingular in the limit $q \to 0$. Equation (89) also has the same scaling dimension as the original local repulsive interaction in (74). Although $V_{dyn}(q)$ does break Lorentz invariance, it does not reduce the order-parameter symmetry. The universality of a Wilson-Fisher fixed point is known to be robust to such space-time symmetry breaking [35,36]. For this reason the critical behavior of the clamped system is unaffected, and thus V_{dyn} can be neglected in the local action.

We have therefore established that the generalized Larkin-Pikin action, following the integration over the Gaussian strain including both thermal and quantum fluctuations, is

$$S[\psi] = S_L[\psi, \tilde{g}, b^*] - \frac{\lambda^2}{2}\left(\frac{1}{K} - \frac{1}{K + \frac{4}{3}\mu}\right) \times \left[\frac{1}{\beta V} \int d^4x \int d^4x' \psi^2(x)\, \psi^2(x')\right] \tag{90}$$

with the local action

$$S_L[\psi, \tilde{g}, b^*] = \int d^4x\, \mathcal{L}_L[\psi, \tilde{g}, b^*] = \int d^4x \left[\frac{1}{2}(\partial_\mu \psi)^2 + \frac{\tilde{g}}{2}\psi^2 + \frac{b^*}{4!}\psi^4\right], \tag{91}$$

where b^* is defined in (88). We note that (90) is a $(d + z)$-dimensional generalization of the effective classical LP action (24), where all space-time configurations are summed to include both thermal and quantum fluctuations. Here z is the dynamical exponent associated with the temporal dimension

since the dispersion $\omega \propto q^z$ leads to $[\xi_\tau] = [\xi]^z$ where ξ_τ and ξ are the correlation time and length, respectively.

B. Generalized Larkin-Pikin equations

The development of the approach is now a simple space-time generalization of its classical counterpart, described in Eqs. (40)–(49). First, we perform a Hubbard-Stratonivich transformation of the space-time-independent interaction in (90):

$$-\frac{\lambda^2}{2}\left(\frac{1}{\kappa}\right)\left[\frac{1}{\beta V}\int d^4x \int d^4x'\, \psi^2(\vec{x})\, \psi^2(\vec{x}')\right] \rightarrow \int d^4x\left[(\lambda\phi)\psi^2(\vec{x}) + \frac{\kappa}{2}\phi^2\right], \tag{92}$$

where

$$\frac{1}{\kappa} = \frac{1}{K} - \frac{1}{K + \frac{4}{3}\mu} \tag{93}$$

is the effective bulk modulus and we have introduced the auxiliary "strain" field ϕ that is space-time independent. Then, we may write

$$\mathcal{Z} = e^{-\tilde{S}(\phi)} = \int \mathcal{D}\psi\; e^{-S[\psi,\phi]}, \tag{94}$$

where $\tilde{S} = \beta\tilde{F}$ and

$$S[\psi,\phi] = \int d^4x\left[\mathcal{L}_L(\psi,\tilde{g}) + \lambda\phi\psi^2 + \frac{\kappa}{2}\phi^2\right] \tag{95}$$

that can be reexpressed as

$$S[\psi,\phi] = \int d^4x[\mathcal{L}_L(\psi,\tilde{g}+2\lambda\phi)] + \frac{\kappa V\beta}{2}\phi^2, \tag{96}$$

where we see that the auxiliary variable ϕ shifts the "mass" (e.g., tuning parameter) of the order parameter ψ by

$$\tilde{g} \rightarrow X = \tilde{g} + 2\lambda\phi. \tag{97}$$

Because the second term in (96) scales as the space-time volume, we can solve for ϕ using a saddle-point evaluation

$$\frac{\partial\tilde{F}[\phi]}{\partial\phi} = 0 \Longrightarrow [\lambda\langle\Psi^2\rangle + \kappa\phi]V = 0, \tag{98}$$

where

$$\Psi^2 \equiv \left[\frac{1}{\beta V}\int d^4x\, \psi^2(x)\right] \tag{99}$$

is the space-time volume average of the energy density and

$$\langle\Psi^2\rangle = \frac{\int d\psi\; \Psi^2 e^{-S_L[\psi]}}{\int d\psi\; e^{-S_L[\psi]}} \tag{100}$$

with S_L as in (91). Equations (98), (99), and (100) lead to

$$\phi = -\frac{\lambda}{\kappa}\langle\Psi^2\rangle. \tag{101}$$

Equations (98)–(101) are identical to their classical counterparts (28)–(37), apart from the replacement of a spatial integral by a space-time integral in (99). The following development, parametrizing the free energy of the clamped and unclamped systems, precisely follows its classical counterpart (41)–(49), but for completeness we include it here in its entirety. The free energy of the clamped system is

$$e^{-\frac{\mathcal{F}(\tilde{g})}{T}} = \int \mathcal{D}[\psi]\, e^{-\mathscr{S}_L[\psi,\tilde{g}]}, \tag{102}$$

where $\mathscr{S}_L$ is defined in (73) and (74) and we have explicitly included its dependence on the tuning parameter $\tilde{g}$. As in the classical case, in writing (102) we have glossed over issues of renormalization. In particular, self-energy corrections to the order-parameter propagators will shift the quantum critical value of g_c from its bare value g_c^0 to a new value $g_c(0)$. All of these renormalization effects can be absorbed into redefinitions of the appropriate variables, in particular, from now on we will redefine $\tilde{g} = g - g_c(0)$.

From (94) and (96) we can write the free energy for our unclamped system as

$$\tilde{\mathcal{F}}[\phi,\tilde{g}] = \mathcal{F}[X] + \frac{\kappa V}{2}\phi^2, \tag{103}$$

where

$$X = \tilde{g} + 2\lambda\phi \tag{104}$$

indicates the shifting the of the tuning parameter due to the presence of energy fluctuations. Now

$$\frac{1}{V}\frac{\partial\mathcal{F}}{\partial X} = \frac{\langle\Psi^2\rangle}{2} \tag{105}$$

so that

$$\phi = -\frac{\lambda\langle\Psi^2\rangle}{\kappa} = -\frac{2\lambda}{V\kappa}\left(\frac{\partial\mathcal{F}}{\partial X}\right) \equiv -\frac{2\lambda}{V\kappa}\mathcal{F}'[X], \tag{106}$$

where we have defined $\mathcal{F}'[X] \equiv (\frac{\partial\mathcal{F}}{\partial X})$ for simplicity. Therefore,

$$\tilde{\mathcal{F}} = \mathcal{F}[X] + \frac{2\lambda^2}{V\kappa}(\mathcal{F}'[X])^2 \tag{107}$$

and

$$X = \tilde{g} - \frac{4\lambda^2}{V\kappa}\mathcal{F}'[X]. \tag{108}$$

Let us define

$$\tilde{f} \equiv \frac{2\lambda}{V\kappa}\tilde{\mathcal{F}}, \quad f \equiv \frac{2\lambda}{V\kappa}\mathcal{F}. \tag{109}$$

Here we recall that the integrals in the action involve an integral over time (72), $\int d^4x = \int_0^\beta d\tau \int d^3x$ where $\beta = \frac{1}{T}$ is a boundary term, so that these free energies are determined at fixed temperature. Therefore, the two equations describing the unclamped system are

$$\tilde{f} = f[X,T] + \lambda(f'[X,T])^2 \tag{110}$$

and

$$\tilde{g} = X + 2\lambda f'[X,T] \tag{111}$$

which have to be solved self-consistently.

Equation (111) can be rewritten as

$$\tilde{g} = X + \frac{2\lambda^2}{\kappa}\langle\Psi^2\rangle_X \tag{112}$$

which leads to

$$\frac{d\tilde{g}}{dX} = 1 - \frac{\lambda^2 V}{\kappa}\chi_{\psi^2}, \tag{113}$$

where

$$\chi_{\psi^2} = \int_0^\beta d\tau \int d^3x \langle \delta\psi^2(\vec{x})\delta\psi^2(0)\rangle \tag{114}$$

is the space-time average of the quantum and thermal "energy" fluctuations. Since $\frac{d\tilde{g}}{dX} = 0$ corresponds to the development of a first-order transition, as previously discussed in the classical case, analogously the generalized LP criterion is

$$\kappa \underset{\sim}{<} \left(\frac{dg_c}{d\ \ln V}\right)^2 \chi_{\psi^2}, \tag{115}$$

where we have assumed

$$g_c[e_{ll}] = g_c - 2\lambda e_{ll} \tag{116}$$

similar to (78). At zero temperature, this expression generalizes the original LP criterion (1) to quantum phase transitions. At finite temperatures, the critical temperature and the critical coupling constant are related by $g_c(T_c) = uT_c^{1/\tilde{\Psi}}$, so that $d\ \ln g_c = \frac{1}{\tilde{\Psi}} d\ \ln T_c$ and the LP criterion becomes

$$\kappa \underset{\sim}{<} \left(\frac{dT_c}{d\ \ln V}\right)^2 \overbrace{\left(\frac{g}{2T_c}\right)^2 \chi_{\psi^2}}^{\Delta C_V/T_c}, \tag{117}$$

where we have identified $\Delta C_v/T_c = (g/2T_c)^2\chi_{\psi^2}$ with the specific-heat capacity. Thus, we see that the generalized Larkin-Pikin equation encompasses the original LP criterion, (1) and also (57), in addition to being applicable at low temperatures. Our next step is to identify a crossover scaling form for the clamped free energy f that includes both thermal and quantum critical fluctuations.

IV. QUANTUM ANNEALING OF THE FIRST-ORDER TRANSITION

A. Amplitude factors

In order to generalize the Larkin-Pikin argument to $T \to 0$, we need to introduce a crossover scaling form for the clamped free energy f in (110) and (111) that is applicable near both the classical and the quantum critical points. The approach we follow here that describes both the quantum and classical cases [27] was adapted from an earlier study used to describe Ising anisotropy at a Heisenberg critical point [37].

At a finite temperature, the location of the phase transition is shifted by the thermal fluctuations, so that

$$g_c(T) = g_c(0) - uT^{\frac{1}{\tilde{\Psi}}}, \tag{118}$$

where $\tilde{\Psi}$ is the shift exponent defined in (71); we note that if the effective dimension of the quantum system is at or below its upper critical dimension $\tilde{\Psi} = \tilde{\nu}z$. For convenience, we will shift the definition of g to absorb the zero-temperature QCP critical coupling constant $g_c(0)$, i.e., $g - g_c(0) \to g$, so that $g_c(T) = -uT^{\frac{1}{\tilde{\Psi}}}$. Now, temperature is a finite-size correction to the quantum critical point, and the free energy is determined

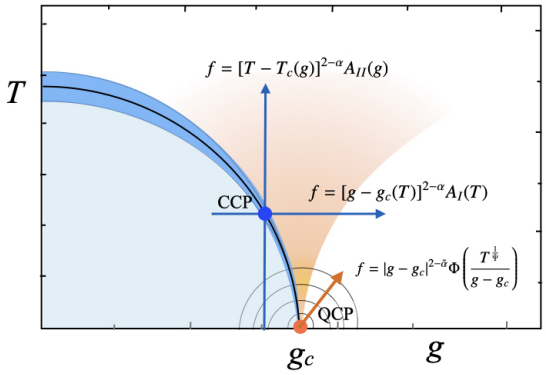

FIG. 5. Schematic showing the dependence of the free energy of the clamped system in the vicinity of the quantum critical point. The scaling function about the QCP determines the amplitude factors for the finite-temperature classical critical point (CCP), given by $A_I(T)$ for a constant temperature sweep and $A_{II}(g)$ for a sweep at constant tuning parameter. Here the location of the quantum critical point at $g_c(0)$ is labeled as simply g_c.

by a crossover function

$$f(g, T) = g^{2-\tilde{\alpha}} \Phi\left(\frac{T^{\frac{1}{\tilde{\Psi}}}}{g}\right) \tag{119}$$

which describes both the quantum critical point and the finite-temperature classical critical point of the clamped system (see Fig. 5), here we will use the convention that an exponent with a tilde refers to the quantum case so that α and $\tilde{\alpha}$ are classical and quantum exponents, respectively. A key point is that at finite temperature, critical behavior now occurs at the shifted value of $g_c(T)$, and the scaling behavior is governed by the finite-temperature critical exponents. Therefore, for a fixed-temperature scan (Fig. 5) for small $g - g_c(T)$,

$$f(g, T) = [g - g_c(T)]^{2-\alpha} A_I(T), \tag{120}$$

where $A_I(T)$ is the amplitude factor for the classical critical point occurring at $g = g_c(T)$. Similarly, if we perform a sweep through the phase transition at constant coupling constant g (Fig. 5), then we can write

$$f[g, T] \sim (T - T_c[g])^{2-\alpha} A_{II}(g), \tag{121}$$

where $A_{II}(g)$ is amplitude factor for the quantum transition at $T_c[g] = (-g/u)^{\tilde{\Psi}}$. The scaling form (119) allows us to determine the form of these amplitude factors (see Appendix C), given by

$$A_I(T) = a_1 T^{\left(\frac{\alpha-\tilde{\alpha}}{\tilde{\Psi}}\right)}, \quad A_{II}(g) = a_2 g^{(1-\tilde{\Psi})(2-\alpha)+(\alpha-\tilde{\alpha})}, \tag{122}$$

where a_1 and a_2 are constants. The resulting expressions for the singular parts of the free energy for constant temperature and constant coupling constant sweeps (see Fig. 5) are

$$f[g, T] = \begin{cases} |g - g_c(T)|^{2-\alpha}\ T^{\frac{\alpha-\tilde{\alpha}}{\tilde{\Psi}}} & \text{(constant } T\text{)}, \\ |T - T_c[g]|^{2-\alpha}\ g^{(1-\tilde{\Psi})(2-\alpha)+(\alpha-\tilde{\alpha})} & \text{(constant } g\text{)}, \end{cases} \tag{123}$$

where, since we are interested in the singular scaling behavior, we have dropped the constants a_1 and a_2.

B. Clausius-Clapeyron relations as $T_c \to 0$

We now examine how the discontinuities $\Delta S(T_c)$ and $\Delta V(T_c)$ in entropy and volume evolve along the first-order phase boundary as the transition T_c is lowered towards zero, and connect them with the Clausius-Clapeyron relation. In this discussion, we shall identify the tuning parameter g with the pressure P, $g \equiv P - P_c$. Using Maxwell's relations we have

$$\frac{dT_c}{dP_c} \equiv \frac{dT_c}{dg_c} = -\left.\frac{\Delta V}{\Delta S}\right|_{T=T_c}. \tag{124}$$

From (118), we have

$$\frac{dT_c}{dg_c} \propto -T_c^{1-\frac{1}{\tilde{\Psi}}}. \tag{125}$$

In the case of particular interest, that of three-dimensional ferroelectrics, the dynamical exponent $z = 1$, so the effective dimension $d_{eff} = 3 + z = 4$ lies at the upper critical dimension. In this case, $\tilde{\Psi} = \tilde{\nu} z = \frac{1}{2}$, we see that this $dT_c/dP_c \propto T_c^{-1}$, implying that $\Delta V/\Delta S$ diverges as $T_c \to 0$. To understand how this happens, we now independently evaluate the temperature dependencies of ΔV and ΔS.

To carry out this calculation, we need to input the quantum-renormalized amplitude factors for the free energy into the parametrized equations (110) and (111). We consider tuning through the first-order phase transition at constant $T = T_c$. The corresponding tuning variable $\mathcal{g} = g - g_c(T_c)$ of the clamped system in (123) is now replaced by the parametric variable X describing the tuning parameter that has been shifted by the long-range interactions $\mathcal{g} \to X$. The singular part of the free energy, by (123), is then

$$f[X] = -|X|^{2-\alpha} T_c^{\frac{\alpha-\tilde{\alpha}}{\tilde{\Psi}}}. \tag{126}$$

Using (110) and (111), the explicit forms of the quantum Larkin-Pikin equations are then

$$\tilde{f}[X] = -|X|^{2-\alpha} T_c^{\frac{\alpha-\tilde{\alpha}}{\tilde{\Psi}}} + \lambda\left[(2-\alpha) X^{1-\alpha} T_c^{\frac{\alpha-\tilde{\alpha}}{\tilde{\Psi}}}\right]^2 \tag{127}$$

and

$$\mathcal{g}[X] = X - 2\lambda(2-\alpha)|X|^{1-\alpha} T_c^{\frac{\alpha-\tilde{\alpha}}{\tilde{\Psi}}} \operatorname{sgn}(X). \tag{128}$$

The only difference between this calculation and the original Larkin-Pikin calculation is the presence of the amplitude factors $T_c^{\frac{\alpha-\tilde{\alpha}}{\tilde{\Psi}}}$. From the original Larkin-Pikin analysis, we know that since $\mathcal{g}[X]$ is a nonmonotonic function of X, the inverse function $X[\mathcal{g}]$ is a discontinuous function of $\mathcal{g}$, given by $X[\mathcal{g}] = \operatorname{sgn}(\mathcal{g}) X_+(|\mathcal{g}|)$. In particular, at $\mathcal{g} = 0$, X jumps from $-X_c$ to $+X_c$, Since the free energy $\tilde{f}[X_c] = \tilde{f}[-X_c]$ is an even function of X, it follows that the first-order transition occurs at $\mathcal{g} = 0$. Using $\mathcal{g}[X_c] = 0$, we obtain

$$X_c = [2\lambda\ (2-\alpha)]^{\frac{1}{\alpha}} T_c^{\frac{\alpha-\tilde{\alpha}}{\alpha\tilde{\Psi}}}. \tag{129}$$

To obtain $\Delta V = d\tilde{f}/d\mathcal{g} = \tilde{f}'[X_c]/\mathcal{g}[X_c]$, we need $\mathcal{g}[X]$ and $\tilde{f}'[X]$. First, we differentiate (128) with respect to X, and using the expression (129) for X_c, we find $\mathcal{g}[X_c] = \alpha$ is just a constant. Also, differentiating (127) with respect to X and substituting (129), we find that

$$\tilde{f}'[X] = -(\alpha/2\lambda) X_c \tag{130}$$

from which we obtain

$$\Delta V(T_c) \propto -T_c^{\frac{\alpha-\tilde{\alpha}}{\alpha\tilde{\Psi}}}. \tag{131}$$

Similarly, to obtain $\Delta S = -d\tilde{f}/dT_c = -\tilde{f}'[X] dX/dT_c$,we need dX/dT_c. Now, since $\mathcal{g} = g + u T_c^{1/\tilde{\Psi}}$ and $\mathcal{g}[X_c] = \alpha$, we obtain

$$\frac{dX}{dT_c} = \frac{1}{\mathcal{g}[X_c]} \frac{d\mathcal{g}}{dT_c} = \frac{u}{\alpha\tilde{\Psi}} T_c^{\frac{1}{\tilde{\Psi}}-1} \tag{132}$$

so that

$$\Delta S[T_c] = -\tilde{f}'[X_c] \frac{dX}{dT_c} \propto T_c^{\frac{\alpha-\tilde{\alpha}}{\alpha\tilde{\Psi}}} T_c^{\frac{1}{\tilde{\Psi}}-1}. \tag{133}$$

For the case $\tilde{\Psi} = \tilde{\nu} z = \frac{1}{2}$, $\alpha = \frac{1}{2}$, $\tilde{\alpha} = 0$, both $\Delta V \sim T_c^2$ and $\Delta S \sim T_c^3$ vanish at absolute zero, but in such a way that their ratio diverges as $T_c \to 0$, in agreement with (125). Naively, the divergence of $\Delta V/\Delta S$ as $T_c \to 0$ might be taken as evidence that the tendency towards a first-order transition increases as the temperature goes to zero, yet the paradox is resolved by noting that ΔS and ΔV simply vanish at different rates, still signifying an approach to a continuous quantum phase transition. More generally, so long as the finite-temperature exponent α exceeds the quantum exponent $\tilde{\alpha}$, $\alpha > \tilde{\alpha}$, (131) indicates that

$$\lim_{T_c \to 0} \Delta V \to 0 \tag{134}$$

so that quite generally, there is no latent work as T_c goes to zero, indicating that quantum fluctuations "anneal" the zero-temperature quantum phase transition to become continuous (see Fig. 6).

V. LARKIN-PIKIN PHASE DIAGRAM

We therefore have a system with a line of classical first-order transitions that ends in a $T = 0$ quantum critical point. Next, we consider application of a field conjugate and parallel and antiparallel to the order parameter. In this section we present the scaling approaches to the critical end points, classical and quantum, and the resulting temperature-field-quantum tuning parameter (g) phase diagram of the generalized Larkin-Pikin problem.

A. Identification of the classical critical end points

We can work out the scaling of the critical end point in the LP mechanism using the scaling form for the free energy

$$f \propto -t^{2-\alpha} \Phi\left(\frac{h}{t^{\beta\delta}}\right), \tag{135}$$

where h is the dimensionless external field, $t = (T - T_c^0)/T_c^0$ is the reduced temperature, and T_c^0 is the transition temperature of the clamped system. We want to know how (135) behaves in a finite field when t is small compared with $h^{1/\beta\delta}$.

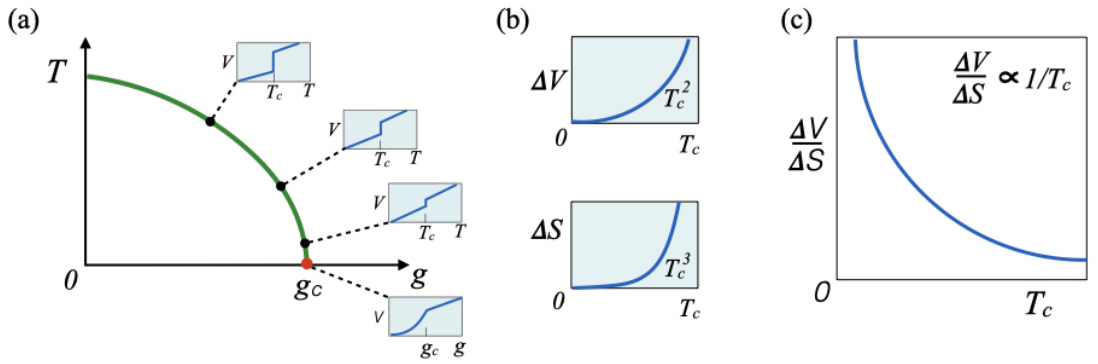

FIG. 6. Schematic figure showing the evolution of the first-order phase transition in the approach to the quantum annealed critical point for the case $\tilde{\Psi} = \tilde{\nu}z = \frac{1}{2}, \alpha = \frac{1}{2}, \tilde{\alpha} = 0$. (a) Evolution of jump in volume, (b) dependence of ΔV and ΔS on T_c, and (c) T_c dependence of $\Delta V/\Delta S$.

In this limit we know that $f \propto h^{\frac{1}{\delta}+1}$ and (135) can be rewritten

$$f \propto -h^{\frac{1}{\delta}+1}\Lambda\Big(\frac{t}{h^{1/\beta\delta}}\Big) = -h^{\frac{2-\alpha}{\beta\delta}}\Lambda\Big(\frac{t}{h^{1/\beta\delta}}\Big), \tag{136}$$

where, using the identity $2-\alpha = \beta(1+\delta)$, we have substituted $(\delta+1)/\delta = (2-\alpha)/\delta\beta$. Comparing (135) and (136), we see that

$$\Lambda = \left(\frac{t^{\beta\delta}}{h}\right)^{1+\frac{1}{\delta}}\Phi. \tag{137}$$

The scaling form of the free energy, defined by (136) and (137), results in the finite-field Larkin-Pikin equations

$$\tilde{f} = h^{1+1/\delta}\Lambda\Big(\frac{X}{h^{1/\beta/\delta}}\Big) + \lambda\Big[h^{\frac{1-\alpha}{\beta\delta}}\Lambda'\Big(\frac{X}{h^{1/\beta\delta}}\Big)\Big]^2,$$
$$t = X - 2\lambda\left[\frac{1}{h^{(\alpha-1)/\beta\delta}}\Lambda'(0) + \frac{X}{h^{\alpha/\beta\delta}}\Lambda''(0)\right], \tag{138}$$

where details of the derivation of (138) are presented in Appendix D.

We recall that criticality of $f[X,h]$ only occurs at $X=0$ ($t=t_c$) indicating that $\tilde{f}[X,h]$ can only be critical at $X=0$. In the region of first-order transitions $t[X]$ is nonmonotonic with two points, a maxima and minima, where the gradient $dt/dX = 0$ goes to zero. As we approach the critical field h_c, the maxima and minima merge together at a point of inflection, meeting at $X=0$. As a result, we deduce that the critical end point occurs when $X=0$ (for criticality) and at $dt/dX = 0$ (merger of maximum and minimum). More succinctly, the critical end point corresponds to the inflection point in $t(X)$. When we impose these two conditions, we can solve for h_c and $t_c = (T_{\rm CEP} - T_c^{(0)})/T_c^{(0)}$, which from (138) implies that

$$h_c = (2\lambda\Lambda'')^{\delta\beta/\alpha} \quad (dt/dX = 0),$$
$$t_c = -\frac{2\lambda\Lambda'}{h_c^{\frac{\alpha-1}{\delta\beta}}} = -h_c^{\frac{1}{\delta\beta}}\frac{\Lambda'}{\Lambda''} = -(2\lambda\Lambda'')^{\frac{1}{\alpha}}\frac{\Lambda'}{\Lambda''} \quad (X=0). \tag{139}$$

We can be sure that these quantities are both positive because $\Delta S = -\partial f[t,h]/\partial t = h^{\frac{1-\alpha}{\beta\delta}}\Lambda'$ is the change in the entropy due to the field, and we expect this to be negative, so that $\Lambda' < 0$ and hence $t_c > 0$. Similarly, $-\frac{\partial^2 f}{\partial t^2} \sim \frac{\Delta C}{T} \sim h^{-\alpha/\delta\beta}\Lambda''$. This quantity gets bigger as the field is reduced, so that $\Lambda'' > 0$, guaranteeing that $h_c > 0$ is real and positive.

B. Field behavior close to the quantum critical end point

When we look at this problem as part of the approach to a QCP, we must now include the amplitude $A_I(T_c) = T_c^{(\alpha-\tilde{\alpha}/\tilde{\Psi})}$. The tuning parameter t of the classical calculation now becomes $\mathcal{g} = g - g_c(T_c) = g - uT_c^{\frac{1}{\tilde{\Psi}}}$. If we now expand around the finite-temperature critical point at a specific T_c, the singular free energy of the clamped system is

$$f[\mathcal{g},h] = -h^{\frac{2-\alpha}{\beta\delta}}\Lambda\Big(\frac{\delta g}{h^{1/\beta\delta}}\Big)A_I(T_c). \tag{140}$$

The Larkin-Pikin equations now become

$$\tilde{f}[X,h] = -h^{\frac{2-\alpha}{\beta\delta}}A_I(T_c)\Lambda\Big(\frac{X}{h^{1/\beta\delta}}\Big) + \lambda\Big[h^{\frac{1-\alpha}{\beta\delta}}A_I(T_c)\Lambda'\Big(\frac{X}{h^{1/\beta\delta}}\Big)\Big]^2,$$
$$\mathcal{g}[X] = X - 2\lambda A_I(T_c)\left[\frac{1}{h^{(\alpha-1)/\beta\delta}}\Lambda'(0) + \frac{X}{h^{\alpha/\beta\delta}}\Lambda''(0)\right], \tag{141}$$

where again $\tilde{f}[X,h]$ refers to the free energy of the unclamped system.

As discussed in the last section, the critical end point occurs at $X=0$. The critical end point is then at $g = -uT_c^{1/\tilde{\Psi}} + \mathcal{g}[0]$. When we do the subsequent algebra, we see that we get similar equations to those we obtained at finite temperature (138) with the replacement $\lambda \to \lambda A_I(T_c)$. Taking Eq. (139) and replacing $t_c \to \mathcal{g}_c$ and $\lambda \to \lambda A_I(T_c)$, we obtain

$$h_c \propto (\lambda A_I(T_c))^{\delta\beta/\alpha} = (\lambda)^{\delta\beta/\alpha}(T_c)^{\frac{\delta\beta(\alpha-\tilde{\alpha})}{\alpha\tilde{\Psi}}},$$
$$\mathcal{g}_c \propto (\lambda A_I(T_c))^{\frac{1}{\alpha}} = \lambda^{\frac{1}{\alpha}}(T_c)^{\frac{(\alpha-\tilde{\alpha})}{\alpha\tilde{\Psi}}}. \tag{142}$$

These equations are valid in the plane of constant T_c. For small λ we can transform these expressions into the plane of constant g, writing $T_c = [g_c(0) - g]^{\tilde{\Psi}}$, while the location of the critical end point is at a temperature $T_{EP} = T_c + \delta T_{EP}$, where

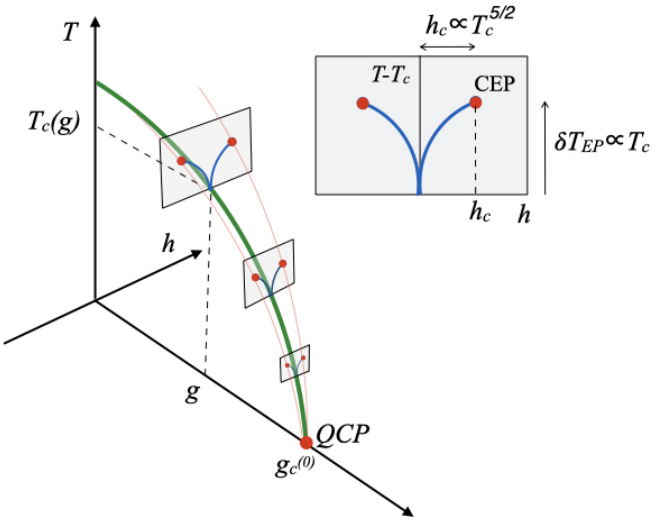

FIG. 7. Evolution of the critical end points with T_c.

$\delta T_{EP} = \mathcal{J}(dT_c/dg) \propto \mathcal{J}(g_c(0) - g)^{\tilde{\Psi}-1}$, which then gives

$$h_c \propto \lambda^{\frac{\delta\beta}{\alpha}} [g_c(0) - g]^{\frac{\delta\beta(\alpha-\tilde{\alpha})}{\alpha}},$$
$$\delta T_{EP} \sim \lambda^{\frac{1}{\alpha}} [g_c(0) - g]^{\frac{\alpha-\tilde{\alpha}}{\alpha}-(1-\tilde{\Psi})}, \quad (143)$$

where we have restored $g_c(0)$. For the Gaussian fixed point considered by Larkin and Pikin, with $\tilde{\alpha} = 0$, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$, $\delta = 5$, $\tilde{\Psi} = \frac{1}{2}$, we have

$$T_c \sim (g_c - g)^{1/2},$$
$$h_c \sim \lambda^{5/2}(g_c - g)^{5/4} \sim T_c^{5/2},$$
$$\delta T_{EP} \sim \lambda^2 (g_c - g)^{1/2}, \quad (144)$$

which yields a pointed, V-shaped "anteater's tongue" as the surface of first-order transitions in the LP problem. In Fig. 7 we present a schematic of the evolution of the critical end points as a function of T_c, and we note that the full Larkin-Pikin phase diagram is displayed in Fig. 1.

VI. IMPLICATIONS FOR OBSERVABLE PROPERTIES

The specific-heat exponent α of the clamped (fixed-volume) system plays a key role in the universality of the classical Larkin-Pikin criterion (1) since the coupling of the order parameter to the lattice is a strain-energy density. For the scalar ($n = 1$) case considered here, $\alpha > 0$, so that $\Delta\kappa$ is singular and the finite-temperature transition is always first order in the unclamped (fixed-pressure) system. By contrast for $d + z > 4$, the system is above its upper critical dimension and there is a continuous transition at $T = 0$ and quantum annealed criticality. The amplitudes of thermodynamic quantities will decrease with temperature in the approach to the quantum critical point, and we have specifically presented this behavior for the latent work and the entropy.

A key motivation for our study has been recent low-temperature experiments on polar insulators that display quantum criticality even though their classical transitions are first order. Many ferroelectrics have scalar order parameters with dynamical exponent $z = 1$, so such three-dimensional materials are in their marginal dimension; logarithmic corrections to the bulk modulus are certainly present but they are not expected to be singular. Indeed, such contributions to the dielectric susceptibility χ in the approach to ferroelectric quantum critical points have not been observed to date [19]; furthermore, here the temperature dependence of χ is described well by a self-consistent Gaussian approach appropriate above its upper critical dimension [19,20]. Therefore, there may be a very weak first-order quantum phase transition [38] but experimentally it appears to be indistinguishable from a continuous one. We note that near quantum criticality the main effect of long-range dipolar interactions, not included in this treatment, is to produce a gap in the longitudinal fluctuations, but the transverse fluctuations remain critical [39–41]; the excellent agreement between theory and experiment at ferroelectric quantum criticality confirms that this is the case [19,20].

Dielectric loss and hysteresis measurements can be used to probe the line of classical first-order transitions, and to determine the nature of the quantum phase transition. The Gruneisen ratio (Γ), the ratio of the thermal expansion and the specific heat, is known to change signs across the quantum phase transition [42,43]; furthermore, it is predicted to diverge at a 3D ferroelectric quantum critical point as $\Gamma \propto \frac{1}{T^2}$ so this would be a good indicator of underlying quantum criticality [20]. Both the bulk modulus and the longitudinal sound velocity should display features near quantum annealed criticality, where specifics are material dependent. Elastic anisotropy may drive this system into an inhomogeneous state [32,34,44]. The coupling of domain dynamics to anisotropic strain has been studied classically for ferroelectrics [45], and implications for the quantum case are a topic for future work.

VII. DISCUSSION AND OPEN QUESTIONS

In summary, we have developed a theoretical framework to describe compressible insulating systems that have classical first-order transitions and display pressure-induced quantum criticality. We have generalized the Larkin-Pikin approach [21] to the quantum case using crossover scaling forms that describe both its classical and its quantum behavior. We show that when the system is above its upper critical dimensionality, there is no latent work at the quantum transition indicating that it is continuous. We then include a field conjugate to the order parameter, and derive the Larkin-Pikin phase diagram with three critical points, two classical and one quantum. Following the original Larkin-Pikin analysis, ours has been performed for a scalar order parameter and isotropic elasticity where the phase transition is first order for all finite temperatures; here, we show that for $d + z > 4$ the quantum transition is continuous. The key point is that a compressible material can host a quantum critical phase even if it displays a first-order transition at ambient pressure. More generally, the order of the classical phase transition can be different from its quantum counterpart.

An interesting question arising from our work is whether the Larkin-Pikin mechanism can be understood in a broader field-theoretic context. The long-range interaction that drives the Larkin-Pikin mechanism relies on the presence of a finite shear modulus: a rigidity of a solid that is absent in a

liquid. The Larkin-Pikin derivation has a topological flavor, in that the $q = 0$ "boundary component" of the strain which drives the long-range interaction, when integrated around a closed loop on a torus, is a topological invariant that counts the number of enclosed defects (19) and which is closely connected with the concept of torsion [46]. In a system with boundaries, we still expect the long-range interaction, but now derived from the boundary waves of the material. There is thus a kind of bulk-boundary correspondence in the phenomenon that may be topological in character. One possibility here is that the Larkin-Pikin interaction, which breaks the Lorentz invariance of the short-range physics, is a kind of symmetry-breaking anomaly [47].

Recently, the possibility of a line of discontinuous transitions ending in a quantum critical point has also been studied in frustrated spin models [48,49], in multiferroics [50,51], and in transition metal difluorides [52]. There are also experiments on metallic systems [53–55] that suggest quantum annealed criticality, so a quantum generalization of the electronic case [56] with possible links to previous work on metallic magnets should be pursued [8]; implications for doped paraelectric materials and polar metals [20] will also be explored. Extension of this work to quantum transitions between two distinct ordered states separated by first-order classical transitions may be relevant to the iron-based superconductors [57] and to the enigmatic heavy-fermion material URu_2Si_2 where quantum critical end points have been suggested [58].

Finally, the possibility of quantum annealed criticality in compressible materials, magnetic and ferroelectric, provides a new setting for the exploration of exotic quantum phases where a broad temperature range can be probed with easily accessible pressures due to the lattice sensitivity of these systems. In particular, the elimination of the Larkin-Pikin mechanism at $T = 0$ exposes a bare quantum critical point, a state of matter with quantum fluctuations on all scales, with the potential for instabilities into novel quantum phases.

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APPENDIX A: GAUSSIAN STRAIN INTEGRATION IN THE CLASSICAL CASE

We would like to integrate out the Gaussian elastic degrees of freedom from the action so that the partition function takes the form

$$Z = \int \mathcal{D}[\psi] \int \mathcal{D}[u] e^{-\mathscr{S}[\psi,u]} \longrightarrow Z = \int \mathcal{D}[\psi] e^{-S[\psi]}, \tag{A1}$$

where the effective action S is a function of the order parameter ψ. We write

$$S[\psi] = S_L[\psi] + \Delta S[\psi], \tag{A2}$$

where

$$S_L[\psi] = \frac{1}{T} \int d^3x \left[\frac{1}{2} (\partial_\mu \psi)^2 + \frac{a}{2} \psi^2 + \frac{b}{4!} \psi^4 \right] \tag{A3}$$

describes the physics of the order parameter in the clamped system with tuning parameter $a \propto \frac{T - T_c}{T_c}$ and $b > 0$ as in (11) and (12). Our task is to calculate the Gaussian integral (21):

$$e^{-\Delta S[\psi]} = \int \mathcal{D}[e_{ab}, u_q] e^{-(\mathcal{S}_E + \mathcal{S}_I)} \tag{A4}$$

with

$$\mathscr{S}_E + \mathscr{S}_I = \frac{1}{T} \int d^3x \left[\frac{1}{2} \left(K - \frac{2}{3} \mu \right) e_{ll}^2(\vec{x}) + \mu e_{ab}(\vec{x})^2 + \lambda \psi^2(\vec{x}) e_{ll}(\vec{x}) \right]. \tag{A5}$$

As discussed in the main text, we separate the strain field into its $q = 0$ and finite-q components (18):

$$e_{ab}(\vec{x}) = e_{ab} + \frac{1}{\sqrt{V}} \sum_{\vec{q} \neq 0} \frac{i}{2} [q_a u_b(\vec{q}) + q_b u_a(\vec{q})] e^{i\vec{q}\cdot\vec{x}}, \tag{A6}$$

so that the action (A5) divides into two terms $\mathscr{S}_E + \mathscr{S}_I = S[e_{ab}, \psi] + S[u, \psi]$. We next define the integrals

$$\int de_{ab} e^{-S[e_{ab},\psi]} = e^{-S_1[\psi]}$$

and

$$\int \mathcal{D}[u] e^{-S[u,\psi]} = e^{-S_2[\psi]} \tag{A7}$$

that treat the $q = 0$ and finite-q elastic contributions to (A4), respectively.

The uniform part of the action is

$$S[e_{ab}, \psi] = \frac{V}{T} \left[\frac{1}{2} \left(K - \frac{2}{3} \mu \right) e_{ll}^2 + \mu e_{ab}^2 \right] + \frac{V}{T} \lambda \psi_{q=0}^2 e_{ll}$$
$$= \frac{1}{2} e_{ab} \mathcal{M}_{abcd} e_{cd} + v_{ab} e_{ab}, \tag{A8}$$

where $\psi^2_{\vec{q}} = \frac{1}{V}\int d^3x\, \psi^2(\vec{x}) e^{i\vec{q}\cdot\vec{x}}$ is the Fourier transform of the fluctuations in "energy density" and

$$\mathcal{M}_{abcd} = \frac{1}{T}\left\{ K\overbrace{(\delta_{ab}\delta_{cd})}^{\mathcal{P}^L_{abcd}} + 2\mu\overbrace{\left(\delta_{ac}\delta_{bd} - \frac{1}{3}\delta_{ab}\delta_{cd}\right)}^{\mathcal{P}^T_{abcd}}\right\}, \quad \text{(A9)}$$

$$v_{ab} = \frac{V}{T}\lambda\psi^2_{q=0}\delta_{ab}. \quad \text{(A10)}$$

In (A9), $\mathcal{P}^L_{abcd}$ and $\mathcal{P}^T_{abcd}$ are independent projection operators ($\mathcal{P}^\Gamma_{abef}\mathcal{P}^\Gamma_{efcd} = \mathcal{P}^\Gamma_{abcd}, \Gamma \in L, T$) associated with the longitudinal and transverse components of the strain.

When we integrate over the uniform part of the strain field (A8),

$$\begin{aligned} S[e_{ab}, \psi] &= \tfrac{1}{2}e_{ab}\mathcal{M}_{abcd}e_{cd} + v_{ab}e_{ab} \;\rightarrow\; S_1[\psi] \\ &= -\tfrac{1}{2}v_{ab}\mathcal{M}^{-1}_{abcd}v_{cd}. \end{aligned} \quad \text{(A11)}$$

Because of the independent nature of the projection operators $\mathcal{P}^{L,T}_{abcd}$ in (A9), we can write the inverse of $\mathcal{M}$ as

$$\mathcal{M}^{-1}_{abcd} = \frac{T}{V}\left[\frac{1}{K}(\delta_{ab}\delta_{cd}) + \frac{1}{2\mu}\left(\delta_{ac}\delta_{bd} - \frac{1}{3}\delta_{ab}\delta_{cd}\right)\right], \quad \text{(A12)}$$

so the Gaussian integral over the uniform part of the strain field yields

$$S_1[\psi] = -\frac{1}{2}v_{ab}\mathcal{M}^{-1}_{abcd}v_{cd} = -\frac{V}{2T}\frac{\lambda^2}{K}\left(\psi^2_{q=0}\right)^2, \quad \text{(A13)}$$

which can also be written as

$$S_1[\psi] = -\frac{\lambda^2}{2T}\left(\frac{1}{K}\right)\left[\frac{1}{V}\int d^3x \int d^3x' \psi^2(\vec{x})\, \psi^2(\vec{x}')\right]. \quad \text{(A14)}$$

The nonuniform part of the action is

$$S[u, \psi] = \frac{1}{T}\sum_{\vec{q}\neq 0}\left(\frac{1}{2}u^*_a(\vec{q})M_{ab}u_b(\vec{q}) + \vec{a}(\vec{q})\cdot\vec{u}(\vec{q})\right), \quad \text{(A15)}$$

where

$$\begin{aligned} M_{ab} &= \left[\left(K - \frac{2}{3}\mu\right)q_aq_b + \mu(q^2\delta_{ab} + q_aq_b)\right], \\ \vec{a}_q &= \left(i\lambda\sqrt{V}\,\psi^2_{-q}\right)\vec{q}. \end{aligned} \quad \text{(A16)}$$

The matrix entering the fluctuating part of the action $S[u, \psi]$ in (A7) can be projected into the longitudinal and transverse components of the strain

$$M_{ab}(\vec{q}) = q^2\left[\left(K + \frac{4}{3}\mu\right)\hat{q}_a\hat{q}_b + \mu(\delta_{ab} - \hat{q}_a\hat{q}_b)\right], \quad \text{(A17)}$$

where $\hat{q}_a = q_a/q$ are the direction cosines of $\vec{q}$. Inversion of this matrix is then

$$M^{-1}_{ab}(\vec{q}) = q^{-2}\left[\left(K + \frac{4}{3}\mu\right)^{-1}\hat{q}_a\hat{q}_b + \mu^{-1}(\delta_{ab} - \hat{q}_a\hat{q}_b)\right], \quad \text{(A18)}$$

so the Gaussian integral over fluctuating part of the strain field leads to

$$\begin{aligned} S[u, \psi] &= \frac{1}{T}\sum_{\vec{q}\neq 0}\frac{1}{2}u^*_a(\vec{q})M_{ab}(\vec{q})u_b(\vec{q}) + \vec{a}(\vec{q})\cdot\vec{u}(\vec{q}) \\ \rightarrow S_2[\psi] &= -\frac{1}{2T}\sum_{\vec{q}\neq 0}a_a(-\vec{q})M^{-1}_{ab}(\vec{q})a_b(\vec{q}) \\ &= -\frac{V}{2T}\sum_{\vec{q}\neq 0}\psi^2_{-q}\psi^2_q\frac{\lambda^2}{K + \frac{4}{3}\mu}. \end{aligned} \quad \text{(A19)}$$

We can rewrite this as a sum over *all* $\vec{q}$, plus a remainder at $\vec{q} = 0$:

$$\begin{aligned} S_2[\psi] &= -\frac{V}{2T}\sum_{\vec{q}}\psi^2_{-q}\psi^2_q\frac{\lambda^2}{K + \frac{4}{3}\mu} + \frac{V}{2T}\left(\psi^2_{q=0}\right)^2\frac{\lambda^2}{K + \frac{4}{3}\mu} \\ &= -\frac{1}{2T}\frac{\lambda^2}{K + \frac{4}{3}\mu}\int d^3x\, \psi^4(\vec{x}) + \frac{V}{2T}\left(\psi^2_{q=0}\right)^2\frac{\lambda^2}{K + \frac{4}{3}\mu} \end{aligned} \quad \text{(A20)}$$

which can be reexpressed as

$$S_2[\psi] = -\frac{\lambda^2}{2T}\left(\frac{1}{K + \frac{4}{3}\mu}\right)\left\{\int d^3x\, \psi^4(\vec{x}) - \left[\frac{1}{V}\int d^3x \int d^3x' \psi^2(\vec{x})\, \psi^2(\vec{x}')\right]\right\}. \quad \text{(A21)}$$

The first term is a local attraction while the second term corresponds to a long-range repulsion.

When we combine (A14) and (A21), we obtain

$$\Delta S[\psi] = -\frac{\lambda^2}{2T}\left\{\left(\frac{1}{K + \frac{4}{3}\mu}\right)\int d^3x\, \psi^4(\vec{x}) + \left(\frac{1}{K} - \frac{1}{K + \frac{4}{3}\mu}\right)\left[\frac{1}{V}\int d^3x \int d^3x' \psi^2(\vec{x})\, \psi^2(\vec{x}')\right]\right\}. \quad \text{(A22)}$$

Recalling (A2), we note that we can group the first term in (A22) in the local $S_L[\psi]$ (A3) to obtain the results

$$S[\psi] = S_L[\psi, a, b^*] - \frac{\lambda^2}{2T}\left(\frac{1}{K} - \frac{1}{K + \frac{4}{3}\mu}\right)\left[\frac{1}{V}\int d^3x \int d^3x' \psi^2(\vec{x})\, \psi^2(\vec{x}')\right], \quad \text{(A23)}$$

where

$$S_L[\psi, a, b^*] = \frac{1}{T}\int d^3x\left[\frac{1}{2}(\partial_\mu\psi)^2 + \frac{a}{2}\psi^2 + \frac{b^*}{4!}\psi^4\right] \quad \text{(A24)}$$

with a renormalized local interaction

$$b^* = b - \frac{12\lambda^2}{K + \frac{4}{3}\mu} \tag{A25}$$

as in the main text [Eqs. (24)–(26)].

APPENDIX B: GAUSSIAN STRAIN INTEGRATION IN THE QUANTUM CASE

We would like to integrate out the Gaussian elastic degrees of freedom so that the partition function takes the form

$$Z = \int \mathcal{D}[\psi] \int \mathcal{D}[u] e^{-\mathscr{S}[\psi,u]} \longrightarrow Z = \int \mathcal{D}[\psi] e^{-S[\psi]}, \tag{B1}$$

where the integrals are over space-time [Eq. (73)], $S = \int d^4x\, L \equiv \int_0^\beta d\tau \int d^3x\, L$. We write the effective action

$$S[\psi] = S_L[\psi] + \Delta S[\psi], \tag{B2}$$

where

$$S_L[\psi] = \int d^4x \left[\frac{1}{2}(\partial_\mu \psi)^2 + \frac{a}{2}\psi^2 + \frac{b}{4!}\psi^4 \right] \tag{B3}$$

and

$$e^{-\Delta S[\psi]} = \int \mathcal{D}[e,u] e^{-(\mathscr{S}_E[u] + \mathscr{S}_I[\psi,e])} \tag{B4}$$

with

$$\mathscr{S}_E + \mathscr{S}_I = \int d^4x \left[\frac{\rho}{2}\dot{u}_l^2 + \left(K - \frac{2}{3}\mu\right) e_{ll}^2(\vec{x}) + \frac{1}{2}2\mu e_{ab}(\vec{x})^2 + \lambda \psi^2(\vec{x}) e_{ll}(\vec{x}) \right]. \tag{B5}$$

This action can be cast into matrix form

$$\mathscr{S}_E + \mathscr{S}_I = \frac{1}{2}\sum_q u_i M_{ij} u_j + \lambda \sum_j u_j \psi_j^2 \to \frac{\lambda^2}{2}\sum_{i,j} \psi_i^2 M_{i,j}^{-1} \psi_j^2, \tag{B6}$$

where now the summations run over the discrete wave vector and Matsubara frequencies $q \equiv (i\nu_n, \vec{q})$, where $\nu_n = \frac{2\pi}{\beta} n$, $\vec{q} = \frac{2\pi}{L}(j,l,k)$. As discussed in the main text, we separate out the static $\vec{q} = 0$ component of the strain tensor (81), writing

$$e_{ab}(x,\tau) = e_{ab} + \frac{1}{\sqrt{V\beta}} \sum_{i\nu_n} \sum_{\vec{q}\neq 0} \frac{i}{2}[q_a u_b(q) + q_b u_b(q)] \times e^{i(\vec{q}\cdot\vec{x} - \nu_n \tau)}. \tag{B7}$$

We note that there is no time dependence in the uniform part of the strain since we restrict ourselves to smooth Gaussian deformations of the solid [see discussion in main text preceding (81)]. However, the fluctuating component includes all Matsubara frequencies; with these caveats, the quantum integration of the strain fields closely follows that of the classical case. Given the form of the elastic tensor (B7), the action (B6) naturally divides into two terms

$$\mathscr{S}_E + \mathscr{S}_I = S[e_{ab}, \psi] + S[u, \psi] \tag{B8}$$

corresponding to the distinct uniform and finite-$\vec{q}$ contributions to the strain, and we define the respective integrals

$$\int de_{ab} e^{-S[e_{ab},\psi]} = e^{-S_1[\psi]}$$

and

$$\int \mathcal{D}[u] e^{-S[u,\psi]} = e^{-S_2[\psi]}, \tag{B9}$$

so that

$$\Delta S[\psi] = S_1[\psi] + S_2[\psi]. \tag{B10}$$

The uniform part of the action

$$\begin{aligned} S[e_{ab}, \psi] &= \int d^4x \left[\frac{1}{2}\left(K - \frac{2}{3}\mu\right) e_{ll}^2 + \frac{1}{2}2\mu e_{ab}^2 \right] + \frac{V}{T}(\lambda \psi_{q=0}^2) e_{ll} \\ &= \frac{1}{2} e_{ab} \mathcal{M}_{abcd} e_{cd} + v_{ab} e_{ab}, \end{aligned} \tag{B11}$$

where

$$\begin{aligned} \mathcal{M}_{abcd} &= \left[K(\delta_{ab}\delta_{cd}) + 2\mu\left(\delta_{ac}\delta_{bd} - \frac{1}{3}\delta_{ab}\delta_{cd}\right)\right], \\ v_{ab} &= V\beta\lambda\psi_{q=0}^2 \delta_{ab} \end{aligned} \tag{B12}$$

is similar to the classical case (A9), but now

$$\psi_q^2 = \frac{1}{V\beta}\int d^4x\, \psi^2(x) e^{-i(\vec{q}\cdot\vec{x} - \nu_n\tau)} \tag{B13}$$

is the space-time Fourier transform of the order-parameter intensity. When we integrate over the uniform part of the strain field, we obtain

$$\begin{aligned} S[e_{ab}, \psi] &= \frac{1}{2} e_{ab} \mathcal{M}_{abcd} e_{cd} + v_{ab} e_{ab} \to S_1[\psi] \\ &= -\frac{1}{2} v_{ab} \mathcal{M}^{-1}_{abcd} v_{cd} \end{aligned} \tag{B14}$$

which, as in the classical case, can be reexpressed as

$$S_1[\psi] = -\frac{\lambda^2 \beta V}{2K}\left(\psi_{q=0}^2\right)^2 \tag{B15}$$

using (A9), (A12), and (B12) where $\beta = \frac{1}{T}$.

The nonuniform part of the elastic contribution to (B4) is

$$S[u,\psi] = \sum_{i\nu_n}\sum_{\vec{q}\neq 0}\left(\frac{1}{2}u_a^*(q) M_{ab} u_b(q) + \vec{a}(q)\cdot\vec{u}(q)\right), \tag{B16}$$

where $q = (\vec{q}, i\nu_n)$ and we use roman letters (e.g., a, b) to denote spatial variables so that q_a is a spatial component of q. Here,

$$\begin{aligned} M_{ab} &= \left[\rho\nu_n^2\left(K - \frac{2}{3}\mu\right) q_a q_b + \mu(q^2\delta_{ab} + q_a q_b)\right], \\ \vec{a}_q &= \left(i\lambda\sqrt{V\beta}\ \psi_{-q}^2\right)\vec{q}. \end{aligned} \tag{B17}$$

This matrix can be projected into its longitudinal and transverse components

$$M_{ab}=\left\{\left[\rho\nu_n^2+\left(K+\frac{4}{3}\mu\right)\right]\hat{q}_a\hat{q}_b+\left(\rho\nu_n^2+\mu\right)(\delta_{ab}-\hat{q}_a\hat{q}_b)\right\}, \tag{B18}$$

where $\hat{q}_a = q_a/q$ is the unit vector. Inversion of this matrix is then

$$M_{ab}^{-1}=\left[\frac{1}{\rho\left(\nu_n^2+c_L^2q^2\right)}\hat{q}_a\hat{q}_b+\frac{1}{\rho\left(\nu_n^2+c_T^2q^2\right)}(\delta_{ab}-\hat{q}_a\hat{q}_b)\right], \tag{B19}$$

where

$$c_L^2=\frac{K+\frac{4}{3}\mu}{\rho},\quad c_T^2=\frac{2\mu}{\rho} \tag{B20}$$

are the longitudinal and transverse sound velocities; the two terms appearing in M^{-1} are recognized as the propagators for longitudinal and transverse phonons.

When we integrate over the fluctuating component of the strain field, only the longitudinal phonons couple to the order parameter:

$$\begin{aligned}&\frac{1}{2}\sum_{i\nu_n}\sum_{\vec{q}\neq 0}u_a^*(q)M_{ab}(q)u_b(q)+\vec{a}(q)\cdot\vec{u}(q)\\&\quad\rightarrow S_2[\psi]=-\frac{1}{2}\sum_{i\nu_n}\sum_{\vec{q}\neq 0}a_a(-q)M_{ab}^{-1}(q)a_b(q)\\&\quad=-\frac{V\beta\lambda^2}{2}\sum_{i\nu_n,\vec{q}\neq 0}\psi_{-q}^2\psi_q^2\left(\frac{q^2}{\rho\nu_n^2+\left(K+\frac{4}{3}\mu\right)q^2}\right).\end{aligned} \tag{B21}$$

In this last term

$$\left(\frac{q^2}{\rho\nu_n^2+\left(K+\frac{4}{3}\mu\right)q^2}\right), \tag{B22}$$

the $\vec{q}=0$ term vanishes for any finite ν_n, but in the case where $\nu_n = 0$, the limiting $\vec{q}\rightarrow 0$ form of this term is finite:

$$\left(\frac{q^2}{\rho\nu_n^2+\left(K+\frac{4}{3}\mu\right)q^2}\right)\Bigg|_{\vec{q}\rightarrow 0}=\begin{cases}0, & \nu_n\neq 0\\ \frac{1}{K+\frac{4}{3}\mu}, & \nu_n=0.\end{cases} \tag{B23}$$

We can thus replace

$$\sum_{i\nu_n,\vec{q}\neq 0}\psi_{-q}^2\psi_q^2\left(\frac{q^2}{\rho\nu_n^2+\left(K+\frac{4}{3}\mu\right)q^2}\right)\rightarrow\sum_{i\nu_n,\vec{q}}\psi_{-q}^2\psi_q^2\left(\frac{q^2}{\rho\nu_n^2+\left(K+\frac{4}{3}\mu\right)q^2}\right)-\frac{\left(\psi_{q=0}^2\right)^2}{K+\frac{4}{3}\mu} \tag{B24}$$

so that

$$S_2[\psi]=\frac{V\beta\lambda^2}{2\left(K+\frac{4}{3}\mu\right)}\left(\psi_{q=0}^2\right)^2-\frac{V\beta\lambda^2}{2}\sum_{i\nu_n,\vec{q}}\psi_{-q}^2\psi_q^2\left(\frac{q^2}{\rho\nu_n^2+\left(K+\frac{4}{3}\mu\right)q^2}\right), \tag{B25}$$

which can be rewritten as

$$S_2[\psi]=\frac{V\beta\lambda^2}{2\left(K+\frac{4}{3}\mu\right)}\left\{\left(\psi_{q=0}^2\right)^2-\sum_{i\nu_n,\vec{q}}\psi_{-q}^2\psi_q^2\left(1-\frac{\nu_n^2/c_L^2}{\nu_n^2/c_L^2+q^2}\right)\right\}, \tag{B26}$$

where c_L is defined in (B20).

If we now combine (B15) and (B26), recalling (B10), we obtain

$$\Delta S=-\frac{V\beta\lambda^2}{2}\left\{\left(\frac{1}{K}-\frac{1}{K+\frac{4}{3}\mu}\right)\left(\psi_{q=0}^2\right)^2--\left(\frac{1}{K+\frac{4}{3}\mu}\right)\sum_{i\nu_n,\vec{q}}\psi_{-q}^2\psi_q^2\left(1-\frac{\nu_n^2/c_L^2}{\nu_n^2/c_L^2+q^2}\right)\right\}. \tag{B27}$$

The useful space-time expression

$$\int d^4x\frac{e^{i(q-q')x}}{\beta V}=\delta_{qq'} \tag{B28}$$

allows us to rewrite (B27) in space-time coordinates as

$$\Delta S=-\frac{\lambda^2}{2}\left\{\frac{1}{\beta V\kappa}\int d^4x\,d^4x'\psi^2(x)\psi^2(x')-\left(\frac{1}{K+\frac{4}{3}\mu}\right)\int d^4x\,[\psi^4(x)]+\int d^4x\,d^4x'\psi^2(x)V_{dyn}(x-x')\psi^2(x')\right\}, \tag{B29}$$

where

$$V_{dyn}(x-x')=\frac{1}{\beta V}\sum_{\vec{q},i\nu_n}\frac{e^{-i(\vec{q}\cdot\vec{x}-\nu_n\tau)}}{\left(K+\frac{4}{3}\mu\right)}\frac{\nu_n^2/c_L^2}{\nu_n^2/c_L^2+q^2} \tag{B30}$$

with

$$\frac{1}{\kappa} = \frac{1}{K} - \frac{1}{K + \frac{4}{3}\mu} \tag{B31}$$

is the effective bulk modulus. Recalling (B2), we note that we can group the second and third terms in (B29) in $S_L[\psi]$ (B3) to obtain the results

$$S[\psi] = S_L[\psi] - \frac{\lambda^2}{2}\left(\frac{1}{K} - \frac{1}{K + \frac{4}{3}\mu}\right)\left[\frac{1}{\beta V}\int d^4x \int d^4x' \psi^2(x)\, \psi^2(x')\right], \tag{B32}$$

with

$$S_L[\psi] = \int d^4x \left[\frac{1}{2}(\partial_\mu \psi)^2 + \frac{a}{2}\psi^2 + \frac{b^*}{4!}\psi^4 + \frac{1}{2}\psi^2(x) V_{dyn}(x - x')\psi^2(x')\right], \tag{B33}$$

where the Fourier transform of $V_{dyn}(x - x')$ is

$$V_{dyn}(q) = \frac{\lambda^2}{K + \frac{4}{3}\mu}\left(\frac{v_n^2/c_L^2}{\vec{q}^2 + v_n^2/c_L^2}\right) \tag{B34}$$

and

$$b^* = b - \frac{12\lambda^2}{K + \frac{4}{3}\mu} \tag{B35}$$

as in the main text [Eqs. (86)–(88)]. We note that (B34) is finite and nonsingular in the limit $q \to 0$, and has the same scaling dimensions as the original local repulsive interaction in (B3); from a scaling perspective it is thus local and will not affect the critical behavior of the clamped system.

APPENDIX C: DERIVATION OF THE AMPLITUDE FACTORS FOR THE CROSSOVER SCALING

In order to generalize the Larkin-Pikin argument to $T \to 0$, we need to introduce a crossover scaling form for the clamped free energy f in (110) and (111) that is applicable near both the classical and the quantum critical points [27]. At a finite temperature, the location of the phase transition is shifted by the thermal fluctuations, so that

$$g_c(T) = g_c(0) - uT^{\frac{1}{\tilde{\Psi}}}, \tag{C1}$$

where $\tilde{\Psi}$ is called the shift exponent; we note that if the effective dimension of the quantum system is below its upper critical dimension $\tilde{\Psi} = \tilde{\nu}z$. For convenience, we will take the zero-temperature QCP critical coupling constant to be zero, $g_c(0) = 0$. Now, temperature is a finite-size correction to the quantum critical point, and the free energy is determined by a crossover function

$$f(g, T) = g^{2-\tilde{\alpha}}\Phi\left(\frac{T^{\frac{1}{\tilde{\Psi}}}}{g}\right) \tag{C2}$$

that describes both the quantum critical point and the finite-temperature classical critical point of the clamped system (see Fig. 5); here, we will use the convention that α and $\tilde{\alpha}$ refer to the classical and quantum exponents, respectively. At a finite temperature the critical behavior now occurs at the shifted value of $g_c(T)$, governed by the finite-temperature specific-heat exponent α. For a fixed-T scan with small $g - g_c(T)$, the singular part of the free energy is

$$f(g, T) = [g - g_c(T)]^{2-\alpha} A_I(T), \tag{C3}$$

where $A_I(T)$ is the amplitude factor for the classical critical point occurring at $T_c = T$.

The scaling form (C2) allows us to determine the form of this amplitude factor. The crucial observation is that the classical critical point occurs at a value $T^{1/\Psi}/g = -1/u$, so that $\Phi(\vec{x})$ must have a singularity of the form $(g + uT^{\frac{1}{\tilde{\Psi}}})^{2-\alpha} = g^{2-\alpha}(1 + u\frac{T^{1/\tilde{\Psi}}}{g})^{2-\alpha} \sim (1 + ux)^{2-\alpha}$, so that the scaling function takes the form

$$\Phi(\vec{x}) = (1 + ux)^{2-\alpha}\tilde{\Phi}(\vec{x}). \tag{C4}$$

To see this in detail, let us rewrite (C2) as

$$\begin{aligned} f(g, T) &= [g - g_c(T)]^{2-\alpha}\frac{g^{2-\tilde{\alpha}}}{[g - g_c(T)]^{2-\alpha}}\Phi\left(\frac{T^{\frac{1}{\tilde{\Psi}}}}{g}\right) \\ &= [g - g_c(T)]^{2-\alpha}\frac{g^{\alpha-\tilde{\alpha}}}{\left(1 + u\frac{T^{\frac{1}{\tilde{\Psi}}}}{g}\right)^{2-\alpha}}\Phi\left(\frac{T^{\frac{1}{\tilde{\Psi}}}}{g}\right). \end{aligned} \tag{C5}$$

In other words,

$$f(g, T) = [g - g_c(T)]^{2-\alpha} g^{\alpha-\tilde{\alpha}}\tilde{\Phi}\left(\frac{T^{\frac{1}{\tilde{\Psi}}}}{g}\right), \tag{C6}$$

where

$$\tilde{\Phi}(\vec{x}) = \frac{\Phi(\vec{x})}{(1 + ux)^{2-\alpha}}. \tag{C7}$$

To ensure a classical phase transition at finite temperature with the right exponent, the crossover function $\tilde{\Phi}$ must be smooth around $x = -1/u$, in other words, the original crossover function contains a hidden singularity at $x = -1/u$ and factorizes as follows:

$$\Phi(\vec{x}) = (1 + ux)^{2-\alpha}\tilde{\Phi}(\vec{x}), \tag{C8}$$

as inferred in (C4). Thus, at finite temperature, the singularity at zero temperature splits into a shifted singularity with modified exponent $2 - \alpha$:

$$f(g, T) = [g - g_c(T)]^{2-\alpha} A[g, T], \tag{C9}$$

where the amplitude factor is given by

$$A[g, T] = g^{\alpha-\tilde{\alpha}}\tilde{\Phi}\left(\frac{T^{\frac{1}{\tilde{\Psi}}}}{g}\right). \tag{C10}$$

Suppose we carry out a sweep at constant temperature T_c (Fig. 5); then, near the classical critical line, we may replace $g = g_c(T_c) = uT_c^{\frac{1}{\Psi}}$, so that $T_c^{1/\Psi}/g = -1/u$ inside the crossover function and

$$f[g, T_c] \sim (g - g_c)^{2-\alpha} A_I(T_c), \tag{C11}$$

where

$$A_I(T_c) = A[g_c(T), T_c] = a_1 T_c^{\left(\frac{\alpha-\tilde{\alpha}}{\Psi}\right)}, \tag{C12}$$

and $a_1 = [u^{\alpha-\tilde{\alpha}}\tilde{\Phi}(-\frac{1}{u})]$. Likewise, if we carry out a sweep through the phase transition at constant g_c (Fig. 5), then we can write

$$f[g_c, T] \sim (T - T_c)^{2-\alpha} A_{II}(g_c), \tag{C13}$$

where

$$\begin{aligned} A_{II}(g_c) &= \left(\frac{dg_c}{dT}\right)^{2-\alpha} g_c^{\alpha-\tilde{\alpha}} \tilde{\Phi}\left(-\frac{1}{u}\right) \\ &= a_2 g_c^{(1-\Psi)(2-\alpha)+(\alpha-\tilde{\alpha})} \end{aligned} \tag{C14}$$

with $a_2 = (\frac{u^{\Psi}}{\tilde{\Psi}})^{2-\alpha}\tilde{\Phi}(-\frac{1}{u})$. Summarizing, the amplitude factors are then

$$A_I(T_c) = a_1 T_c^{\left(\frac{\alpha-\tilde{\alpha}}{\Psi}\right)}, \quad A_{II}(g_c) = a_2 g_c^{(1-\Psi)(2-\alpha)+(\alpha-\tilde{\alpha})} \tag{C15}$$

as given in (122) in the main text.

APPENDIX D: DERIVATION OF THE LARKIN-PIKIN EQUATIONS IN FINITE FIELD

We use our scaling form for the free energy

$$f \propto -t^{2-\alpha}\Phi\left(\frac{h}{t^{\beta\delta}}\right), \tag{D1}$$

where h is the dimensionless external field, $t = (T - T_c^0)/T_c^0$ is the reduced temperature, and T_c^0 is the transition temperature of the clamped system. We want to describe the behavior of (D1) when t is small compared with $h^{1/\beta\delta}$. In this limit we know that $f \propto h^{\frac{1}{\delta}+1}$ and (D1) can be rewritten

$$f \propto -h^{\frac{1}{\delta}+1}\Lambda\left(\frac{t}{h^{1/\beta\delta}}\right) = -h^{\frac{2-\alpha}{\beta\delta}}\Lambda\left(\frac{t}{h^{1/\beta\delta}}\right), \tag{D2}$$

where, using the identity $2 - \alpha = \beta(1 + \delta)$, we have substituted $(\delta + 1)/\delta = (2 - \alpha)/\delta\beta$. Comparing (D1) and (D2), we obtain

$$\Lambda = \left(\frac{t^{\beta\delta}}{h}\right)^{1+\frac{1}{\delta}}\Phi. \tag{D3}$$

If $y = \frac{t}{h^{1/\beta\delta}}$ and $z = y^{-\beta\delta}$, then we have

$$\Lambda(y) = y^{\beta(1+\delta)}\Phi(y^{-\beta\delta}) = y^{2-\alpha}\Phi(y^{-\beta\delta}). \tag{D4}$$

We note that at large values of $z = h/t^{\beta\delta}$, small values of $y = \frac{t}{h^{1/\beta\delta}}$, the free energy can be expanded perturbatively in y around $y = 0$, so that

$$f[X, h] = -h^{\frac{2-\alpha}{\beta\delta}}\left[\Lambda(0) + \frac{X}{h^{1/\beta\delta}}\Lambda'(0) + \frac{X^2}{2h^{2/\beta\delta}}\Lambda''(0)\right], \tag{D5}$$

where we have replaced t by the parametrized variable X of the unclamped material. This means that

$$\frac{\partial f}{\partial X} \equiv f'_X = -\left[\frac{1}{h^{(\alpha-1)/\beta\delta}}\Lambda'(0) + \frac{X}{h^{\alpha/\beta\delta}}\Lambda''(0)\right]. \tag{D6}$$

When (D5) and (D6) are input into the LP equations (48) and (49), the Larkin-Pikin equations in a finite field become

$$\begin{aligned} \tilde{f} &= h^{1+1/\delta}\Lambda\left(\frac{X}{h^{1/\beta/\delta}}\right) + \lambda\left[h^{\frac{1-\alpha}{\beta\delta}}\Lambda'\left(\frac{X}{h^{1/\beta\delta}}\right)\right]^2, \\ t &= X - 2\lambda\left[\frac{1}{h^{(\alpha-1)/\beta\delta}}\Lambda'(0) + \frac{X}{h^{\alpha/\beta\delta}}\Lambda''(0)\right], \end{aligned} \tag{D7}$$

which are exactly the equations (138) in the main text.

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