

## Emergent Critical Phase and Ricci Flow in a 2D Frustrated Heisenberg Model

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We introduce a two-dimensional frustrated Heisenberg antiferromagnet on interpenetrating honeycomb and triangular lattices. Classically the two sublattices decouple, and “order from disorder” drives them into a coplanar state. Applying Friedan’s geometric approach to nonlinear sigma models, we obtain the scaling of the spin stiffnesses governed by the Ricci flow of a four-dimensional metric tensor. At low temperatures, the relative phase between the spins on the two sublattices is described by a six-state clock model with an emergent critical phase.

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A remarkable discovery of recent years is that frustrated two-dimensional Heisenberg models can evade the Hohenberg-Mermin-Wagner theorem [1,2] via the development of long-range discrete order driven by short-range thermal spin fluctuations: such discrete long-range order develops despite the persistence of a finite spin correlation length, leading to a finite temperature Ising ( $\mathbb{Z}_2$ ) or  $\mathbb{Z}_3$  Potts phase transition [3–9]. This phenomenon is well established in the  $J_1$ - $J_2$  Heisenberg model on the square lattice and has recently been realized in iron-based superconductors [10]. An interesting question motivated by this discovery is whether it can be generalized to higher  $\mathbb{Z}_p$  ( $p \geq 5$ ) order. If one can show, in addition, that these emergent discrete degrees of freedom are described by a  $p$ -state clock model [11,12], the unique situation arises that a Heisenberg spin system exhibits two Berezinskii-Kosterlitz-Thouless (BKT) transitions which bracket a critical phase. In a system of discrete Ising spins, such a scenario was reported to occur on the triangular lattice [13,14].

In this Letter, we introduce such a Heisenberg model defined on interpenetrating honeycomb and triangular lattices [Fig. 1(a)] with nearest-neighbor antiferromagnetic coupling. This model may be realized with cold spinful atoms in optical lattices, where it arises naturally in the limit of large on-site interactions [15–18]. Another promising experimental route is to employ STM techniques for nanofabrication and spin-resolved read-out of stacked triangular and honeycomb monolayers of magnetic atoms like Cr or Co [19–22]. For classical spins the two sublattices are decoupled giving rise to a  $\text{SO}(3) \times \text{O}(3)/\text{O}(2)$  order parameter. “Order from disorder” [23,24] drives the two sublattices into a coplanar spin configuration [25] with a  $\text{SO}(3) \times \text{U}(1)$  order parameter and a sixfold in-plane potential. In the coplanar state we explicitly show that the  $\text{U}(1)$  degrees of freedom decouple to form an emergent  $\mathbb{Z}_6$  clock model with an intermediate power-law phase.

This nontrivial decoupling of the  $\text{U}(1)$  phase is essential for the critical phase to occur.

A novel aspect of our work is that we apply Friedan’s coordinate-independent approach to nonlinear sigma models [26] to the scaling of the spin stiffness. In this approach the configurations of the two-dimensional (2D) spin system correspond to a worldsheet of a string evolving in four dimensions, where the metric is determined by the components of the antiferromagnetic stiffness and its renormalization corresponds to a Ricci flow of the metric tensor. We also note that the decoupling of our  $\text{U}(1)$  phase can be viewed as a toy model for the compactification of a four-dimensional (4D) string theory.

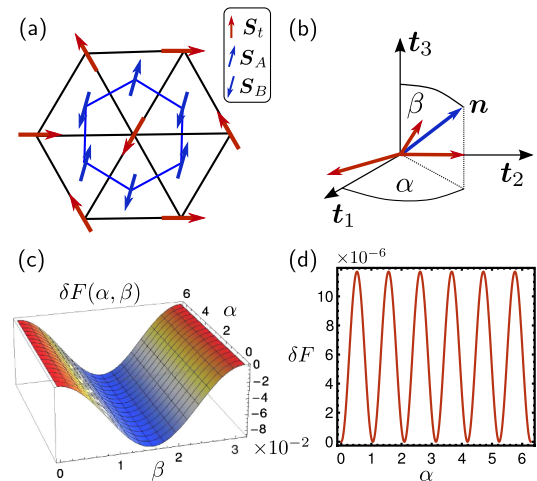


FIG. 1 (color online). (a) Heisenberg model on the “windmill” lattice. (b) Definition of angles  $\alpha$  and  $\beta$  describing the relative orientation of magnetic order on triangular and honeycomb lattice. (c) and (d) Angle dependent free energy correction  $\delta F$  from thermal and quantum spin fluctuations for parameters  $J_{hh} = J_{tt} = 1$ ,  $J_{th} = 0.4$ ,  $T = 1$ . Panel (d) is for fixed  $\beta = \pi/2$ .

Specifically we study the antiferromagnetic Heisenberg model on a decorated 2D triangular lattice [cf. Fig. 1(a)]; the associated Bravais lattice has three basis sites per unit cell at positions  $\mathbf{b}_t = a_0(0, 2/\sqrt{3})$ ,  $\mathbf{b}_A = (0, 0)$ , and  $\mathbf{b}_B = a_0(0, 1/\sqrt{3})$ , where indices  $A, B$  label the two honeycomb sites. We set the lattice constant  $a_0 = 1$ . The Hamiltonian is  $H = H_{tt} + H_{AB} + H_{tA} + H_{tB}$  with

$$H_{ab} = J_{ab} \sum_{j=1}^{N_L} \sum_{\{\delta_{ab}\}} \mathbf{S}_a(r_j) \cdot \mathbf{S}_b(r_j + \delta_{ab}), \quad (1)$$

where  $\mathbf{S}_a(r_j)$  denote spin operators at Bravais lattice site  $j$  and basis site  $a \in \{t, A, B\}$ . The vectors  $\{\delta_{ab}\}$  point between nearest neighbors of sublattices  $a, b$ . We assume in the following that the spin exchange couplings within the same sublattice are larger than the intersublattice coupling  $J_{th} < J_{tt}, J_{hh}$ , where  $J_{th} \equiv J_{tA} = J_{tB}$  and  $J_{hh} \equiv J_{AB}$ . For decoupled lattices  $J_{th} = 0$ , the classical ground state on the bipartite honeycomb lattice is the usual Néel state, while spins on the triangular lattice arrange in a  $120^\circ$  configuration [27]. Although the exchange fields between the two sublattices exactly cancel for this configuration even for  $J_{th} > 0$ , quantum and thermal fluctuations depend on the relative orientation of the magnetization on the two sublattices. The uniaxial magnetic order on the honeycomb lattice is described by a normal vector  $\mathbf{n}(x)$ , which points along the magnetization on sublattice  $A$ . The biaxial order on the triangular lattice is characterized by a triad of orthonormal vectors  $\{\mathbf{t}_j(x)\}$  with  $j = 1, 2, 3$ . Equivalently, it may be expressed by an orthogonal matrix  $t = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3) \in \text{SO}(3)$ . We take the vectors  $\mathbf{t}_{1,2}$  to span the plane of the magnetization on the triangular lattice. The relative order between the two sublattices is thus determined by two angles  $\alpha$  and  $\beta$ , which are defined in Fig. 1(b).

Symmetry considerations dictate the form of the long-wavelength action which takes the form of a nonlinear sigma model (NLSM)

$$S = \int d^2x \left( \frac{K}{2} (\partial_\mu \mathbf{n})^2 + \sum_{j=1}^3 \frac{K_j}{2} (\partial_\mu \mathbf{t}_j)^2 \right) + S_c. \quad (2)$$

The action contains the usual gradient terms of the  $\text{O}(3)/\text{O}(2)$  and the  $\text{SO}(3)$  NLSM for the order parameter on the honeycomb and triangular lattice. The bare spin stiffnesses  $K, K_j$  can be derived in a  $1/S$ -expansion and read  $K = 2J_{hh}S^2/T$ ,  $K_1 = K_2 = \sqrt{3}J_{tt}S^2/4T$ , and  $K_3 = 0$  [28–30]. In addition, the action in Eq. (2) contains two potential terms, generated by short-wavelength spin fluctuations (“order from disorder”) [23,24]

$$S_c = \frac{1}{2} \int d^2x [\gamma \cos^2 \beta + \lambda \sin^6 \beta \sin^2(3\alpha)]. \quad (3)$$

A positive  $\gamma > 0$  favors coplanarity, whereas  $\gamma < 0$  induces  $\mathbf{n}$  to be perpendicular to the plane of the triangular magnetization. The sixfold anisotropy term  $\lambda$  is relevant only for  $\gamma > 0$ .

Heuristically, we expect  $\gamma > 0$ , favoring coplanarity: spins on the honeycomb lattice can minimize their energy by aligning themselves perpendicular to the fluctuation Weiss field from the triangular lattice [24]. To confirm this reasoning, we have performed a Holstein-Primakov spin wave analysis of Eq. (1). Our results for the fluctuation correction to the free energy for arbitrary angles  $\alpha$  and  $\beta$  between the two sublattices are given in Figs. 1(c) and 1(d) and show that  $\gamma > 0$ . For small  $J_{th}$  we find that  $\gamma = (J_{th}/\bar{J})^2 A_\gamma(J_{tt}/J_{hh}, \bar{J}/T)$  is the dominant term in the potential, while  $\lambda = (J_{th}/\bar{J})^6 A_\lambda(J_{tt}/J_{hh}, \bar{J}/T)$ , where  $\bar{J} = \sqrt{J_{tt}J_{hh}}$  and the  $A_{\gamma,\lambda}$  are functions that depend weakly on  $J_{tt}/J_{hh}$ .

As temperature is reduced, the two sublattices enter a coplanar regime. The temperature scale for this crossover is easily determined from standard scaling arguments and yields the coplanar crossover temperature [see Fig. 2(a)]

$$T_{\text{cp}} \approx \frac{J_{hh} S^2}{1 + \ln(1/\gamma)/4\pi} \quad (4)$$

in the case where  $J_{hh} < J_{tt}$ . In the opposite regime  $J_{tt} < J_{hh}$  we obtain an implicit expression for  $T_{\text{cp}}$  that also approaches zero only logarithmically for  $\gamma \rightarrow 0$ . The crossover temperature in Eq. (4) follows from the known flow equation  $\frac{d}{dl} K = -1/2\pi$  for the stiffness  $K$  with running cutoff  $\Lambda(l) = a_0^{-1} e^{-l}$  and the flow of the coplanar potential amplitude  $\gamma(l) = \gamma \exp(2l)$  that is determined by its engineering dimension. While the spin stiffnesses are

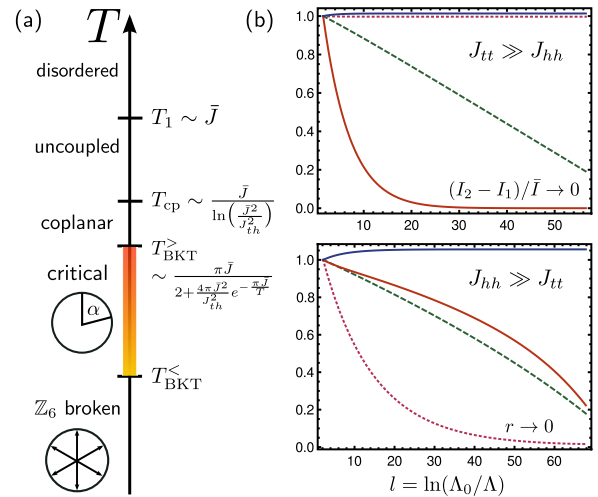


FIG. 2 (color online). (a) Schematic phase diagram. (b) Coplanar RG flow of the variables  $I'_\alpha$  (blue, increasing),  $\bar{I} = (I_1 I_2 I_3)^{1/3}$  (green dashed),  $(I_2 - I_1)/\bar{I}$  (red), and  $r$  (pink dotted). Curves are normalized to initial values at  $l_\gamma$ . Upper panel is for  $J_{tt} \gg J_{hh}$  with  $J_{hh} = 1$ ,  $J_{tt} = 5$ ,  $J_{th} = 0.4$ ,  $T = 0.6$ , and initial values  $\bar{I} = 5.3$ ,  $(I_2 - I_1)/\bar{I} = 0.27$ ,  $r = 0.82$ ,  $I'_\alpha = 1.2$ . Decoupling is due to  $(I_1 - I_2)/\bar{I} \rightarrow 0$ . Lower panel is for  $J_{hh} \gg J_{tt}$  with  $J_{hh} = 5$ ,  $J_{tt} = 1$ ,  $J_{th} = 0.4$ ,  $T = 0.5$ , and initial values  $\bar{I} = 4.5$ ,  $(I_2 - I_1)/\bar{I} = 2.1$ ,  $r = 0.11$ ,  $I'_\alpha = 1.1$ . Decoupling is due to  $r \rightarrow 0$ .

reduced at longer length scales, the potential term grows, and scaling stops when  $\gamma(l_\gamma) = 1$ , which defines a length scale  $a_\gamma = a_0 e^{l_\gamma} \simeq a_0 (J_{hh}/J_{th})^2$ . The coplanar crossover takes place when this length scale is comparable to the shorter of the magnetic correlation lengths on the two sublattices. From the known flow equation of the O(3)/O(2) and the SO(3) NLSM it further follows that the stiffnesses of the triangular lattice approach an isotropic fixed point [31]. The sixfold symmetric potential  $\propto \lambda$  flows to larger values, yet due to  $\lambda \ll \gamma$  it holds that  $\lambda(l_\gamma) \simeq \mathcal{O}(J_{th}^4) \ll 1$ .

Once the two sublattices are coplanar, their dynamics are intimately connected. To describe this regime we impose a hard-core constraint:  $\mathbf{n} \perp \mathbf{t}_3$ , i.e.,  $\beta = \pi/2$ . It is now convenient to introduce a second triad  $\mathbf{h}_{1,2,3}$ , defining a SO(3) matrix  $h = (\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3)$  that describes the magnetic order on the honeycomb lattice with  $\mathbf{h}_1 = \mathbf{n}$ . The coplanar constraint is expressed as  $t = hU$  where  $U = \exp(i\alpha\tau_3)$  determines the relative in-plane orientation

of the two sublattices, defined by the angle  $\alpha$ . We describe  $h$  in terms of three Euler angles,  $h = e^{-i\phi\tau_3} e^{-i\theta\tau_1} e^{-i\psi\tau_3}$ . Here, the  $\tau_a$  satisfy the SU(2) algebra  $[\tau_a, \tau_b] = i\epsilon_{abc}\tau_c$  and take the adjoint form  $(\tau_a)_{bc} = i\epsilon_{bac}$ . The coplanar system is thus determined by a SO(3)  $\times$  U(1) order parameter, defined by three Euler angles and a single relative phase  $\alpha$ .

To analyze this coupled problem we write the action in the form  $S = S_X + S_c$ , where

$$S_X = \frac{1}{2} \int d^2x g_{ij}[X(x)] \partial_\mu X^i(x) \partial_\mu X^j(x) \quad (5)$$

with coordinates  $X = (\phi, \theta, \psi, \alpha)$  and stiffness tensor

$$g = \begin{pmatrix} g^{\text{SO}(3)} & \mathcal{K}^T \\ \mathcal{K} & I_\alpha \end{pmatrix}, \quad (6)$$

where

$$g_{ij}^{\text{SO}(3)} = \begin{pmatrix} (I_1 s_\psi^2 + I_2 c_\psi^2) s_\theta^2 + I_3 c_\theta^2 & (I_1 - I_2) c_\psi s_\theta s_\psi & I_3 c_\theta \\ (I_1 - I_2) c_\psi s_\theta s_\psi & I_1 c_\psi^2 + I_2 s_\psi^2 & 0 \\ I_3 c_\theta & 0 & I_3 \end{pmatrix}$$

with  $s_{Xj} = \sin X^j$  and  $c_{Xj} = \cos X^j$ . In our system we find  $I_1 = K_1 + K_3$ ,  $I_2 = K_1 + K_3 + K$ ,  $I_3 = 2K_1 + K$ , which are set by the stiffnesses of the two sublattices at  $l = l_\gamma$ . The U(1) degree of freedom  $\alpha$  has an initial stiffness  $I_\alpha = 2K_1(l_\gamma)$  and is coupled to the non-Abelian SO(3) sector by the term  $\mathcal{K} = \frac{\kappa}{2}(c_\theta, 0, 1)$  in the four-dimensional metric, where  $\kappa = 4K_1(l_\gamma)$ . The sixfold potential  $S_c(\beta = \frac{\pi}{2}) = \frac{1}{2} \lambda \int d^2x \sin^2(3\alpha)$  is a small but relevant perturbation to  $S_X$ . At length scales where  $\lambda$  is small, the anisotropy  $S_c$  and the gradient term  $S_X$  [Eq. (5)] is the action of a classical string in a four-dimensional space with coordinates  $X(x)$  at the two-dimensional worldsheet point  $x$ , with metric tensor  $g_{ij}[X]$ . Under coordinate transformations  $X_i \rightarrow X'_i$ ,  $S_X$  in Eq. (5) is invariant, with transformed metric  $g'_{lm} = g_{ij} \frac{\partial X^i}{\partial X'^l} \frac{\partial X^j}{\partial X'^m}$ . Like Einstein's theory of gravity, this covariance tells us that the long-wavelength action  $S_X$  is coordinate independent and only depends on the geometric aspects of the mapping  $X(x)$  of the worldsheet to the compact four-dimensional space of the coordinate  $X$ . The renormalization group (RG) flow of the metric tensor must also be covariant under coordinate transformations, and following the geometric formulation of the NLSM by Friedan [26], to two loop order takes the form

$$\frac{dg_{ij}}{dl} = \frac{1}{2\pi} R_{ij} - \frac{1}{8\pi^2} R_i{}^{klm} R_{jklm}, \quad (7)$$

where  $R^{iklm}$  is the Riemann curvature tensor and  $R_{ij} = R^k{}_{ikj}$  is the Ricci tensor [32]. This expression defines a generalized Ricci flow [33]. The Riemann tensor is

determined by the Christoffel symbols  $\Gamma_{jk}^i = \frac{1}{2} g^{il} (g_{jl,k} + g_{kl,j} - g_{jk,l})$  as  $R^k{}_{lij} = \Gamma_{lj,i}^k - \Gamma_{li,j}^k + \Gamma_{ni}^k \Gamma_{lj}^n - \Gamma_{nj}^k \Gamma_{li}^n$ . The flow equations of our five coupling constants  $I_j$ ,  $I_\alpha$ , and  $\kappa$  follow from Eq. (7).

A key insight into the low energy phase diagram is obtained by noting that the coupling term  $\mathcal{K}$  can be eliminated via a coordinate transformation  $\psi \rightarrow \psi' = \psi + r\alpha$  with  $r = \kappa/2I_3$ . This yields a metric  $g$  in Eq. (6) with  $\mathcal{K} = 0$ ,  $I_\alpha \rightarrow I'_\alpha = I_\alpha - \kappa^2/4I_3$  yet with  $g^{\text{SO}(3)}$  that depends on the U(1) phase  $\alpha$  via the above shift of the Euler angle  $\psi$ . This gauge transformation to the appropriate center of mass coordinates allows for clear criteria when the U(1) sector of the theory decouples from the SO(3) sector: if either  $|I_1 - I_2| \ll \sqrt{I_1 I_2}$  or  $r \ll 1$  it follows that  $g^{\text{SO}(3)}$  becomes independent of  $\alpha$  and the U(1) phase decouples from the dynamics of the noncollinear magnetic degrees of freedom. The first criterion follows from the fact that  $g^{\text{SO}(3)}$  is independent of  $\psi$  if  $I_1 = I_2$ , while the second criterion implies that the shift in  $\psi$  is negligible. From Eq. (7) it follows after a lengthy but straightforward calculation that  $I_{1,2,3}$  flow to an isotropic fixed point, while the dimensionless variable  $r$  follows the flow equation (for simplicity we only list the one loop result; the two loop correction does not change our conclusions):

$$\frac{dr}{dl} = -r \frac{(I_1 - I_2)^2}{4\pi I_1 I_2 I_3}. \quad (8)$$

Thus, if the initial anisotropy  $|I_1 - I_2| = K$  is weak, which happens for  $J_{hh} \ll J_{tt}$ , the coupling  $r$  does not change

much. The SO(3) sector, however, quickly becomes isotropic in the 1–2 plane leading to a decoupling of the U(1) phase. On the other hand, in the limit of strong anisotropy for  $J_{hh} \gg J_{tt}$ , where  $|I_1 - I_2|$  is not small, we find that  $r$  vanishes rapidly. In both cases it follows that the phase angle  $\alpha$  emerges as an independent degree of freedom. The  $\beta$ -function for the reduced phase stiffness  $I'_\alpha = I_\alpha - \kappa^2/4I_3$  follows from Eq. (7) as

$$\frac{dI'_\alpha}{dl} = \beta_\alpha = \frac{(I_1 - I_2)^2 r^2}{4\pi I_1 I_2}, \quad (9)$$

and does, as expected, approach zero once either of the two decoupling conditions are fulfilled. Thus, perturbatively no renormalization of the stiffness  $I'_\alpha$  takes place. In Fig. 2(b) we present the coplanar RG flow for two different sets of parameters corresponding to weak and strong initial anisotropy. An interesting aspect of the decoupling follows from the Ricci scalar  $R = g^{ij}R_{ji}$ :

$$R = R^{\text{SO}(3)} - \frac{1}{2\pi I'_\alpha} \beta_\alpha, \quad (10)$$

where  $R^{\text{SO}(3)} = \sum_{j=1}^3 (I_j^{-1} - \frac{1}{2I_1 I_2 I_3} I_j^2)$  is the Ricci scalar of the SO(3) sector. Once the decoupling takes place,  $\beta_\alpha \rightarrow 0$  and the U(1) sector becomes flat. On the other hand  $R \rightarrow R^{\text{SO}(3)}$  grows under renormalization since the stiffnesses  $I_j$  decrease. Thus, we arrive at a flat one-dimensional sector weakly coupled to a three-dimensional manifold with large curvature. This “curling-up” and asymptotic decoupling of a subspace may serve as a toy model for compactification.

Since the decoupling emerges rapidly in both limits  $J_{hh} \ll J_{tt}$  and  $J_{hh} \gg J_{tt}$ , we find that  $\lambda$ , whose flow is governed by  $\frac{d}{dl}\lambda = (2 - 9/\pi I'_\alpha)\lambda$ , is still small at the decoupling lengthscale. The resulting low-energy theory corresponds to  $S = S_{\text{SO}(3)} + S_{\mathbb{Z}_6}$  with

$$S_{\mathbb{Z}_6} = \frac{1}{2} \int d^2x [(I'_\alpha (\partial_\mu \alpha)^2 + \lambda \sin^2(3\alpha))]. \quad (11)$$

This is the well-known six-state clock model that exhibits two consecutive BKT transitions [11]: one at  $T_{\text{BKT}}^>$  that separates a high temperature disordered phase from a low temperature critical phase, where correlations  $\langle \exp\{i[\alpha(x) - \alpha(x')]\} \rangle$  decay with a power law in  $|x - x'|$ , and a second at  $T_{\text{BKT}}^<$  where the  $\mathbb{Z}_6$  symmetry is spontaneously broken, leading to true long-range order with  $\alpha = n\pi/3$  ( $n \in \{1, \dots, 6\}$ ). It is crucial that the decoupling of the U(1) phase occurs first, otherwise the SO(3) sector would screen the long-range interactions between topological defects—vortices at  $T_{\text{BKT}}^>$  or domain walls at  $T_{\text{BKT}}^<$  that are responsible for the BKT transitions and the intermediate critical phase.

Following the RG program of the BKT problem for Eq. (11) we need to take into account that the size of the vortex is now given by the coplanar length scale  $a_\gamma \gg a_0$  [34,35]. We determine the vortex unbinding transition temperature  $T_{\text{BKT}}^>$  implicitly via

$$I'_\alpha (T_{\text{BKT}}^>)^{-1} = \frac{\pi}{2 + 4\pi y (T_{\text{BKT}}^>)} \quad (12)$$

with fugacity  $y = e^{-S_c} a_\gamma^2 / a_0^2$  and core action  $S_c \approx \pi\{1 + \min(K, K_r)\}$ . From Eq. (12) we predict that  $T_{\text{BKT}}^> \lesssim T_{\text{cp}}$ , i.e., the BKT transition is only numerically smaller than the coplanar crossover temperature. The system enters the critical phase soon after it becomes coplanar. Similarly, it follows from Ref. [11] that  $T_{\text{BKT}}^<$  and  $T_{\text{BKT}}^>$  are of the same order of magnitude. The resulting phase diagram is shown in Fig. 2(a).

In summary we have presented a 2D Heisenberg model on a decorated triangular lattice where short wavelength thermal fluctuations select long-range  $\mathbb{Z}_6$  order. This is preceded in temperature by an emergent critical phase that is framed by two BKT transitions. We have written the action of this model as a classical 4D string theory where the spin stiffness is determined by the metric tensor of the manifold; the scaling equations are then extracted as components of the resulting Ricci flow. We note that the decoupling of the U(1) degree of freedom corresponds to a dimensional reduction of the analogous string theory and thus to a toy model of compactification. Finally we note that the emergence of massless modes in collective mode massive theories could have interesting implications for two-dimensional field theories.

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*Note added.*—After obtaining these results we learned of two recent studies: one on a Kitaev-Heisenberg model, where an emergent  $\mathbb{Z}_6$ -symmetry results from a conceptually different mechanism [36], a second on itinerant systems where an emergent  $\mathbb{Z}_4$  Potts model appears [37].

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