Two-frequency mutual coherence function of electromagnetic waves in random media: a path-integral variational solution

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By use of path-integral methods, a general expression is obtained for the two-frequency, two-position mutual coherence function of an electromagnetic pulse propagating through turbulent atmosphere. This expression is valid for arbitrary models of refractive-index fluctuations, wide band pulses, and turbulence of arbitrary strength. The approach presented in this paper was examined in the cases of plane-wave, spherical wave, and Gaussian beam propagation in power-law turbulence and compared with existing numerical and exact results. A number of new results were obtained for the Gaussian beam pulse. Expressions derived here should be applicable to a wide range of practical pulse propagation problems.

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1. INTRODUCTION

The problem of pulse propagation through the turbulent atmosphere is of great interest in many applications, such as ground-to-satellite high-data-rate communication links and lidar operation. The two-frequency mutual coherence function (MCF) is necessary to obtain such pulse propagation characteristics as the instantaneous intensity of the pulse and the coherence bandwidth; in general, it provides a complete description of the second moment of the field.

In weak turbulence the Rytov approximation has been used by many authors to obtain an analytical expression for the MCF with the Kolmogorov spectrum of refractive-index fluctuations or modifications thereof (for a recent application to Gaussian beam pulses, see Refs. 1 and 2).

The theory of pulse propagation in strongly turbulent media has also been investigated by many researchers using a variety of analytical and computational techniques. In particular, an assumption of the Gaussian spectrum of refractive-index fluctuations leads to a quadratic structure function when the spatial separation of observation points in a plane perpendicular to the direction of propagation is small compared with the width of the Gaussian.[3,4]

The quadratic problem has a closed-form analytic solution: Sreenivasiah *et al.*[4,5] solved the parabolic equation satisfied by the two-frequency MCF under the Markov approximation.[6] Their solution is valid for wideband pulses and in the strong fluctuation regime. A narrowband approximation to this result is obtained when the original equation is expanded in the wave-number difference Δ in the beginning, and the resulting system of equations solved via Green's function techniques.[7]

Path integrals provide an especially elegant and powerful method for finding the two-frequency, two-position MCF. Once the parabolic (small-angle scattering) approximation is made, path-integral formulation is possible; in the quadratic case the problem becomes isomorphic to that for a particle in the harmonic-oscillator potential. This path integral is solved in many standard texts.[8,9]

Dashen[10] showed the solution for the one-position MCF in the case of a spherical wave (point source). Rose and Besieris provided a systematic functional integral treatment of Nth-order multifrequency coherence functions for the quadratic turbulence structure function.[11,12] Their approach results in closed-form expressions for multifrequency, multiposition coherence functions of any order, with the deterministic background represented by a polynomial of up to second degree. More recently, Gozani[13] derived a formal path-integral solution to the explicit time-domain second-moment equation applicable to ultrashort pulsed beams.

It is well known that the assumption of Gaussian spectrum of refractive-index fluctuations may lead to some conceptual difficulties and the absence of important qualitative effects.[14,15]

However, the problem of finding analytical expressions for a multifrequency, multiposition MCF in the case of the Kolmogorov spectrum of refractive-index fluctuations has proven more elusive. Neither the path integral nor the corresponding parabolic equation can be solved exactly; only approximate, numeric, or asymptotic expressions can be derived. Numerical results for the two-frequency MCF with the Kolmogorov spectrum were obtained by Sreenivasiah,[3] Lee and Jokipii,[16] and others. Furutsu[17] solved the irradiance space–time transport equation for a plane-wave pulse and obtained convergent series representations for the two-frequency MCF and the instantaneous intensity of the pulse. More recently, Oz and Heyman[18–22] used the modal expansion method to solve the parabolic equation for the two-frequency MCF. They found approximate solutions for the plane wave, the point source, and the beam wave (for the one-position MCF only).

Finally, Fante[23] used the extended Huygens–Fresnel (HF) principle to find the two-position, two-frequency MCF in a turbulent medium.

In this paper we use a path-integral variational approach[8] for the evaluation of two-frequency, two-position MCFs with general power-law turbulence structure functions. The rest of the paper is organized as follows. Section 2 is devoted to the development of the formal path-integral solution for the two-frequency MCF. In Section 3 we derive exact plane-wave, spherical wave, and Gaussian beam MCFs for the quadratic spectrum of refractive-index fluctuations. These results, although already available in the literature,[4,5,11,12] will form the starting point for our variational approach. In Section 4 we obtain a general variational expression for the MCF with arbitrary power-law structure functions. In Section 5 we apply this expression to the point-source, the plane-wave, and the Gaussian beam pulse propagation in Kolmogorov turbulence; we compare some of our results with those available from other authors, including calculations based on the use of the extended HF principle.[23] The expressions we obtain contain an integral over the initial source distribution, which we solve numerically for the plane-wave and the Gaussian beam cases. Finally, in Section 6 we briefly summarize our conclusions and discuss future applications of the path-integral variational technique.

2. PATH-INTEGRAL APPROACH TO THE NORMALIZED MUTUAL COHERENCE FUNCTION WITH A GENERAL POWER-LAW STRUCTURE FUNCTION

Consider an electromagnetic wave propagating in a random medium along the z axis; using the parabolic (quasi-optical) approximation and neglecting depolarization and backscatter, we obtain the following scalar equation for a wave amplitude[6,24]:

$$2ik\frac{\partial u(\mathbf{r}, z, k)}{\partial z} + \Delta u(\mathbf{r}, z, k) + k^2\tilde{\epsilon}u(\mathbf{r}, z, k) = 0, \tag{1}$$

where $k = \omega\langle\epsilon\rangle^{1/2}/c$ is the wave number; ω is the wave frequency; $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$; $\mathbf{r} = (x, y)$ denotes location in the plane perpendicular to the direction z of wave propagation; $u(\mathbf{r}, z, k)$ is an electric field component with the free-space phase factored out: $E(\mathbf{r}, z, t) = u(\mathbf{r}, z, k)\exp(ikz - i\omega t)$; and $\tilde{\epsilon} = (\epsilon - \langle\epsilon\rangle)/\langle\epsilon\rangle$, where ϵ is the dielectric permittivity and $\langle\epsilon\rangle$ is its average over random medium fluctuations. For simplicity, we restrict ourselves to the case of homogeneous and nondispersive background medium.

The wave amplitude is subject to the initial condition

$$u(\mathbf{r}, z = 0, k) = u^0(\mathbf{r}).$$

Wave equation (1) has the following functional integral solution[25,26]

$$u(\mathbf{R}, \mathcal{Z}, k) = \int \mathrm{d}^2\mathbf{R}_0 u^0(\mathbf{R}_0)\int \mathcal{D}^2\mathbf{s}\delta[\mathbf{s}(0) - \mathbf{R}_0]\delta[\mathbf{s}(\mathcal{Z}) - \mathbf{R}] \times \exp\left\{\frac{ik}{2}\int_0^{\mathcal{Z}}\mathrm{d}z\left[\left(\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}z}\right)^2 + \tilde{\epsilon}(\mathbf{s}, z)\right]\right\}. \tag{2}$$

We start by computing a spherical wave MCF; the plane-wave and beam wave MCFs will be obtained from it by integration over the initial source distribution (assumed to be at $z = 0$). We define the normalized spherical wave MCF as follows (all z labels will be dropped for brevity from now on):

$$\Gamma_p^N(\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_1^0, \mathbf{R}_2^0; k_1, k_2) \equiv \frac{\langle u(\mathbf{R}_1, \mathbf{R}_1^0, k_1)u^*(\mathbf{R}_2, \mathbf{R}_2^0, k_2)\rangle}{u^f(\mathbf{R}_1, \mathbf{R}_1^0, k_1)u^{f*}(\mathbf{R}_2, \mathbf{R}_2^0, k_2)}, \tag{3}$$

where

$$u^f(\mathbf{R}, \mathbf{R}^0, k) = -\frac{ik}{2\pi\mathcal{Z}}\exp\left(\frac{ik}{2\mathcal{Z}}|\mathbf{R} - \mathbf{R}^0|^2\right)$$

is a solution to the free-space parabolic wave equation.

Use of the Markov approximation[6] for dielectric permittivity fluctuations in homogeneous and isotropic media amounts to asserting that

$$\langle\tilde{\epsilon}(\mathbf{r}, z)\tilde{\epsilon}(\mathbf{r}', z')\rangle = \delta(z - z')A(|\mathbf{r} - \mathbf{r}'|, z).$$

In all calculations below, we assume for simplicity that the transverse correlation function A is not explicitly dependent on z or the wave number (within the bandwidth of interest). Then we obtain for the normalized MCF in Eq. (3):

$$\Gamma_p^N = \exp\left[-\frac{(k_1 - k_2)^2\mathcal{Z}A(0)}{8}\right]\frac{I\{A_p\}}{I\{0\}}, \tag{4}$$

where

$$I\{A_p\} = \int_f \mathcal{D}^2\mathbf{s}_1\int_f \mathcal{D}^2\mathbf{s}_2 \exp\left\{\int_0^{\mathcal{Z}}\mathrm{d}z\left[\frac{ik_1}{2}\dot{\mathbf{s}}_1^2(z) - \frac{ik_2}{2}\dot{\mathbf{s}}_2^2(z) + \frac{k_1k_2}{4}A_p|\mathbf{s}_1(z) - \mathbf{s}_2(z)|^p\right]\right\}, \tag{5}$$

$\dot{} \equiv \mathrm{d}/\mathrm{d}z$, and $\int_f\mathcal{D}^2\mathbf{s}$ implies path integration with end points fixed at prespecified positions by appropriate δ functions [cf. Eq. (2)]. Note that we restricted ourselves to the power-law correlation function expansion in Eqs. (4) and (5):

$$A(\mathbf{s}) \simeq A(0) + A_p|\mathbf{s}|^p, \qquad \text{with } 1 < p \leqslant 2.$$

$p = 5/3$ corresponds to Kolmogorov turbulence, and $p = 2$ represents the Gaussian spectrum of refractive-index fluctuations.

Using the orthogonal transformation of variables[10]

$$\mathbf{v} = \mathbf{s}_1 - \mathbf{s}_2,$$

$$\mathbf{u} = (k_1\mathbf{s}_1 - k_2\mathbf{s}_2)/(k_1 - k_2),$$

we obtain for Eq. (5):

$$I\{A_p\} \equiv I_u\{0\}I_v\{A_p\} = \int_f \mathcal{D}^2\mathbf{u} \exp\left[\frac{i}{2}(k_1 - k_2)\int_0^{\mathcal{Z}} \mathrm{d}z \dot{\mathbf{u}}^2\right]\int_f \mathcal{D}^2\mathbf{v} \times \exp\left[\int_0^{\mathcal{Z}} \mathrm{d}z\left(-\frac{i}{2}\frac{k_1k_2}{k_1 - k_2}\dot{\mathbf{v}}^2 + \frac{k_1k_2}{4}A_p|\mathbf{v}|^p\right)\right]. \tag{6}$$

Therefore

$$\Gamma_p^N(\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_1^0, \mathbf{R}_2^0; k_0, \Delta) = \exp\left[-\frac{\Delta^2 \mathcal{Z} A(0)}{8}\right]\frac{I_v\{A_p\}}{I_v\{0\}}, \tag{7}$$

where

$$\frac{I_v\{A_p\}}{I_v\{0\}} = \frac{\int_f \mathcal{D}^2\mathbf{v} \exp\left[\int_0^{\mathcal{Z}} \mathrm{d}z\left(-\frac{i}{2}\frac{k_1k_2}{k_1 - k_2}\dot{\mathbf{v}}^2 + \frac{k_1k_2}{4}A_p|\mathbf{v}|^p\right)\right]}{\int_f \mathcal{D}^2\mathbf{v} \exp\left[\int_0^{\mathcal{Z}} \mathrm{d}z\left(-\frac{i}{2}\frac{k_1k_2}{k_1 - k_2}\dot{\mathbf{v}}^2\right)\right]}. \tag{8}$$

We introduce the difference and the average of the wave numbers through

$$k_0 = (k_1 + k_2)/2,$$

$$\Delta = k_1 - k_2.$$

We remark here that one can expand all paths subject to the imposed boundary conditions as follows:

$$\mathbf{v}(z) = \frac{z}{\mathcal{Z}}(\mathbf{R}_1 - \mathbf{R}_2) + \left(1 - \frac{z}{\mathcal{Z}}\right)(\mathbf{R}_1^0 - \mathbf{R}_2^0) + \sum_{n=1}^{\infty} \mathbf{a}_n \sin\left(n\pi\frac{z}{\mathcal{Z}}\right). \tag{9}$$

If the last term on the right-hand side of Eq. (9) is neglected, the path integrals cancel out in the expression for Γ_p^N:

$$\Gamma_p^N(\mathbf{R}_1 - \mathbf{R}_2, \mathbf{R}_1^0 - \mathbf{R}_2^0; k_0, \Delta) = \exp\left[-\frac{\Delta^2 \mathcal{Z} A(0)}{8}\right]\exp\left[\frac{k_1k_2}{4}A_p \times \int_0^{\mathcal{Z}} \mathrm{d}z\left|\frac{z}{\mathcal{Z}}(\mathbf{R}_1 - \mathbf{R}_2) + \left(1 - \frac{z}{\mathcal{Z}}\right)(\mathbf{R}_1^0 - \mathbf{R}_2^0)\right|^p\right]. \tag{10}$$

Equation (10) leads to the extended HF principle formulation for the two-frequency, two-position MCF.[23] The omission of all propagation path deviations from the straight line, however, is a heuristic approximation that cannot be justified under arbitrary conditions.[26,27]

3. EXACT SOLUTIONS FOR QUADRATIC TURBULENCE

In this section we shall find exact analytical expressions for plane-wave, spherical wave, and Gaussian beam two-frequency MCFs in the case of the quadratic structure function ($p = 2$).

We start by taking the path integral in the expression for $I_v\{A_2\}$. We note in passing that one way to find the $\int_f \mathcal{D}^2\mathbf{v}\ldots$ path integral would be to use the expansion (9) in it, leading to the replacement of $\int_f \mathcal{D}^2\mathbf{v}$ by $\Pi_{n=1}^{\infty}\int \mathrm{d}^2\mathbf{a}_n$.[8,10] However, once we recognize that the path integral in question is isomorphic to that of a two-dimensional harmonic oscillator, we can write down the expression for $I_v\{A_2\}$ at once using standard solutions provided in Refs. 8 and 9:

$$I_v\{A\} = \left(\frac{ik_1k_2}{2\pi\Delta\mathcal{Z}}\right)\frac{\eta}{\sin\eta}\exp\left\{\left(-\frac{ik_1k_2}{2\Delta\mathcal{Z}}\right)\frac{\eta}{\sin\eta} \times [(\boldsymbol{\xi}^2 + \mathbf{p}^2)\cos\eta - 2\boldsymbol{\xi}\mathbf{p}]\right\}. \tag{11}$$

Then the normalized quadratic MCF is

$$\Gamma_2^N(\mathbf{p}, \boldsymbol{\xi}; k_0, \Delta) = \exp\left[-\frac{\Delta^2\mathcal{Z}A(0)}{8}\right]\frac{\eta}{\sin\eta}\exp\left[\frac{ik_1k_2}{2\Delta\mathcal{Z}}(\boldsymbol{\xi} - \mathbf{p})^2\right] \times \exp\left\{\left(-\frac{ik_1k_2}{2\Delta\mathcal{Z}}\right)\frac{\eta}{\sin\eta}[(\boldsymbol{\xi}^2 + \mathbf{p}^2)\cos\eta - 2\boldsymbol{\xi}\mathbf{p}]\right\}. \tag{12}$$

We used the following notation in Eqs. (11) and (12):

$$\mathbf{R} = (\mathbf{R}_1 + \mathbf{R}_2)/2, \qquad \mathbf{p} = \mathbf{R}_1 - \mathbf{R}_2,$$

$$\mathbf{R}^0 = (\mathbf{R}_1^0 + \mathbf{R}_2^0)/2, \qquad \boldsymbol{\xi} = \mathbf{R}_1^0 - \mathbf{R}_2^0,$$

and $\eta = \mathcal{Z}[-(i/2)A_2\Delta]^{1/2}$.

With this result, we are in a position to compute spherical wave, plane-wave, and Gaussian beam quadratic MCFs (the expressions for which were previously obtained in Refs. 4, 5, 11, and 12). In particular, we immediately obtain for two point sources located at $\mathbf{R}_1^0$ and $\mathbf{R}_2^0$

$$\Gamma_2^{\mathrm{sph}}(\mathbf{R}, \mathbf{R}^0, \mathbf{p}, \boldsymbol{\xi}; k_0, \Delta) = u^f(\mathbf{R}_1, \mathbf{R}_1^0, k_1)u^{f*}(\mathbf{R}_2, \mathbf{R}_2^0, k_2) \times \Gamma_2^N(\mathbf{R}, \mathbf{R}^0, \mathbf{p}, \boldsymbol{\xi}; k_0, \Delta)$$
$$= \frac{k_0^2 - \Delta^2/4}{(2\pi\mathcal{Z})^2}\frac{\eta}{\sin\eta}\exp\left[-\frac{\Delta^2\mathcal{Z}A(0)}{8}\right] \times \exp\left[\frac{i\Delta}{2\mathcal{Z}}(\mathbf{R} - \mathbf{R}^0)^2\right]\exp\left[\frac{ik_0}{\mathcal{Z}}(\mathbf{R} - \mathbf{R}^0)(\mathbf{p} - \boldsymbol{\xi})\right] \times \exp\left[\frac{ik_0^2}{2\Delta\mathcal{Z}}(\mathbf{p} - \boldsymbol{\xi})^2\right]\exp\left\{-\frac{i}{2}\frac{k_0^2 - \Delta^2/4}{\Delta\mathcal{Z}}\frac{\eta}{\sin\eta} \times [(\boldsymbol{\xi}^2 + \mathbf{p}^2)\cos\eta - 2\boldsymbol{\xi}\mathbf{p}]\right\}. \tag{13}$$

Furthermore, we derive for a unit-amplitude plane wave

$$\Gamma_2^{\mathrm{pl}}(\mathbf{p};\,k_0,\,\Delta) = \int \mathrm{d}^2R^0 \int \mathrm{d}^2\xi\,\Gamma_2^{\mathrm{sph}}(\mathbf{R},\,\mathbf{R}^0,\,\mathbf{p},\,\boldsymbol{\xi};\,k_0,\,\Delta)$$
$$= \exp\left[-\frac{\Delta^2 \mathcal{Z} A(0)}{8}\right]\frac{1}{\cos\eta} \times \exp\left[\frac{i}{2}\,\frac{(k_0^2 - \Delta^2/4)\,\eta}{\Delta \mathcal{Z}}\tan\eta\;\mathbf{p}^2\right]. \quad (14)$$

Note that there is no $\mathbf{R}$ dependence in the plane-wave case.

Finally, for a Gaussian beam of unit amplitude at its center we obtain

$$\Gamma_2^{\mathrm{Gaus}}(\mathbf{R},\,\mathbf{p};\,k_0,\,\Delta) = \int \mathrm{d}^2R^0 \int \mathrm{d}^2\xi \exp\left[-\left(\frac{1}{W_0^2} + i\,\frac{k_1}{2F_0}\right)|\mathbf{R}_1^0|^2\right] \times \exp\left[-\left(\frac{1}{W_0^2} - i\,\frac{k_2}{2F_0}\right)|\mathbf{R}_2^0|^2\right] \times \Gamma_2^{\mathrm{sph}}(\mathbf{R},\,\mathbf{R}^0,\,\mathbf{p},\,\boldsymbol{\xi};\,k_0,\,\Delta), \quad (15)$$

where W_0 is the beam size at $\mathcal{Z} = 0$ and F_0 is the phase-front radius of curvature (both are assumed for simplicity to be independent of the wave number).

Taking the initial source distribution integrals, we easily derive

$$\Gamma_2^{\mathrm{Gaus}}(\mathbf{R},\,\mathbf{p};\,k_0,\,\Delta) = \frac{k_1k_2}{4\mathcal{Z}^2}\,\frac{\eta}{\sin\eta}\exp\left[-\frac{\Delta^2\mathcal{Z}A(0)}{8}\right]\left(\frac{ik_1k_2}{2\Delta\mathcal{Z}}\,\alpha\eta\cot\eta + \frac{\alpha\beta}{4} + \frac{k_0^2}{4\tilde{F}^2}\right)^{-1}$$
$$\times \exp\left(\frac{i\Delta}{2\mathcal{Z}}\mathcal{R}^2\right)\exp\left(\frac{ik_0}{\mathcal{Z}}\mathbf{R}\cdot\mathbf{p}\right)\exp\left(\frac{ik_0^2}{2\Delta\mathcal{Z}}\mathbf{p}^2\right)\exp\left[\frac{(\Delta\mathbf{R} + k_0\mathbf{p})^2}{4\alpha\mathcal{Z}^2}\right]\exp\left(-\frac{ik_1k_2}{2\Delta\mathcal{Z}}\,\eta\cot\eta\;\mathbf{p}^2\right)$$
$$\times \exp\left\{-\frac{\left[\dfrac{k_1k_2}{\Delta\mathcal{Z}}\,\dfrac{\eta}{\sin\eta}\mathbf{p} - \dfrac{k_0^2}{\Delta\mathcal{Z}}\mathbf{p} - \dfrac{k_0}{\mathcal{Z}}\mathbf{R} + \dfrac{ik_0}{2\alpha\tilde{F}\mathcal{Z}}(\Delta\mathbf{R} + k_0\mathbf{p})\right]^2}{\left(\dfrac{2ik_1k_2}{\Delta\mathcal{Z}}\,\eta\cot\eta + \beta + \dfrac{k_0^2}{\alpha\tilde{F}^2}\right)}\right\}, \quad (16)$$

where

$$1/\tilde{F} = 1/F_0 - 1/\mathcal{Z},$$

$$\alpha = 2/W_0^2 + i\Delta/2\tilde{F},$$

$$\beta = 2/W_0^2 + i(\Delta/2F_0 - 2k_0^2/\Delta\mathcal{Z}).$$

Some of the important simplified cases include a beam focused at $\mathcal{Z}$ ($1/\tilde{F} = 0$) and a plane wave ($1/F_0 = 0$, $1/W_0^2 = 0$). In the latter case, the general expression (16) reduces to Eq. (14).

4. VARIATIONAL APPROACH TO PATH INTEGRALS FOR GENERAL POWER-LAW STRUCTURE FUNCTIONS

In this section we derive a generalized expression for the normalized spherical wave MCF, applicable to the arbitrary power-law spectrum of refractive-index fluctuations. We employ a path-integral variational approach originally developed in Ref. 8 for the problem of electrons in a polar crystal.

We start by analytically continuing expressions for Γ_2^N and Γ_p^N according to the following rules:

$$z = i\bar{z},$$

$$A_p = -i\bar{A}_p\,.$$

This implies that

$$\eta = i\bar{\mathcal{Z}}\left(-\frac{\bar{A}_2\Delta}{2}\right)^{1/2} \equiv i\bar{\eta}.$$

We observe that since the plane-wave coherence bandwidth is given by $k_{2,\mathrm{coh}} = -8/(\bar{A}_2\bar{\mathcal{Z}}^2)$ in the Gaussian fluctuations case,[5,18] $\bar{\eta} = 2(\Delta/k_{2,\mathrm{coh}})^{1/2}$. This variable arises naturally as a result of reducing the differential equation for the two-frequency plane-wave MCF to its universal form.[5]

Consequently, we obtain for $I_v\{A_p\}$ (for simplicity, we shall drop all overbars used in the analytical continuation from now on)

$$I_v\{A_p\} = \int_f \mathcal{D}^2\mathbf{v}\exp\left(-\int_0^{\mathcal{Z}}\mathrm{d}z\,\mathcal{L}_p\right), \quad (17)$$

where

$$\mathcal{L}_p = \frac{k_1k_2}{2(k_1 - k_2)}\dot{\mathbf{v}}^2 - \frac{k_1k_2}{4}A_p|\mathbf{v}|^2. \quad (18)$$

We can specify $k_1 > k_2$ here (or $\Delta > 0$) without any loss of generality.

Now consider

$$\frac{\Gamma_p^N}{\Gamma_2^N} = \frac{\displaystyle\int_f \mathcal{D}^2\mathbf{v}\exp\left(-\int_0^{\mathcal{Z}} \mathrm{d}z\,\mathcal{L}_2\right)\exp\left[-\frac{k_1k_2}{4}\int_0^{\mathcal{Z}} \mathrm{d}z(A_2|\mathbf{v}|^2 - A_p|\mathbf{v}|^p)\right]}{\displaystyle\left[\int_f \mathcal{D}^2\mathbf{v}\exp\left(-\int_0^{\mathcal{Z}} \mathrm{d}z\,\mathcal{L}_2\right)\right]}$$

$$\equiv \left\langle \exp\left[-\frac{k_1k_2}{4}\int_0^{\mathcal{Z}} \mathrm{d}z(A_2|\mathbf{v}|^2 - A_p|\mathbf{v}|^p)\right]\right\rangle, \quad (19)$$

where the $\langle \ldots \rangle$ symbol stands for a normalized expectation value with respect to $\mathcal{L}_2$.

The variational principle states that

$$\exp(J) \equiv \left\langle \exp\left[-\frac{k_1k_2}{4}\int_0^{\mathcal{Z}} \mathrm{d}z(A_2|\mathbf{v}|^2 - A_p|\mathbf{v}|^p)\right]\right\rangle \geqslant \exp\left[\left\langle -\frac{k_1k_2}{4}\int_0^{\mathcal{Z}} \mathrm{d}z(A_2|\mathbf{v}|^2 - A_p|\mathbf{v}|^p)\right\rangle\right]. \quad (20)$$

This statement is akin to a simple one-dimensional inequality $\langle e^x \rangle \geqslant e^{\langle x \rangle}$.

Therefore we found an estimate for the power-law normalized spherical wave MCF:

$$\Gamma_p^N \geqslant \Gamma_2^N \exp(J). \quad (21)$$

We find the best approximation to Γ_p^N by maximizing the right-hand side of relation (21) with respect to the free parameters of the Gaussian problem, which include the strength of the quadratic turbulence and the width of the dielectric permittivity fluctuation spectrum.

In what follows, we derive an analytical expression for J using the Fourier expansion of the correlation function $A(\rho)$ in isotropic turbulence:

$$A_p\rho^p \simeq (2\pi)\int \mathrm{d}^2\kappa\,\Phi_p(\kappa)[\exp(i\boldsymbol{\kappa\rho}) - 1]. \quad (22)$$

After substituting Fourier expansion (22) into the expression for J, one obtains the following path integral in its numerator:

$$J_{\text{num}} \equiv \left(\frac{\pi}{2}\right)k_1k_2\int_f \mathcal{D}^2\mathbf{v}\exp\left(-\int_0^{\mathcal{Z}} \mathrm{d}z\,\mathcal{L}_2\right)\int \mathrm{d}^2\kappa\,\Psi(\kappa) \times \int_0^{\mathcal{Z}} \mathrm{d}z'\left\{\exp\left[i\boldsymbol{\kappa}\int_0^{\mathcal{Z}} \mathrm{d}z\,\mathbf{v}(z)\delta(z - z')\right] - 1\right\}, \quad (23)$$

where $\Psi(\kappa) = \Phi_p(\kappa) - \Phi_2(\kappa)$.

We note that the functional integral in Eq. (23) is isomorphic to that for the harmonic oscillator under the influence of the external force (minus the free quadratic Lagrangian for which the solution was obtained in the previous sections). Taking this integral,[8,9] we immediately see that

$$J_{\text{num}} = \left(\frac{k_1k_2}{2}\right)^2 \frac{\eta}{\Delta\mathcal{Z}\sinh\eta}\exp\left\{-\frac{k_1k_2}{2\Delta\mathcal{Z}}\frac{\eta}{\sinh\eta} \times [(\boldsymbol{\xi}^2 + \mathbf{p}^2)\cosh\eta - 2\boldsymbol{\xi}\mathbf{p}]\right\} \times \int_0^{\mathcal{Z}} \mathrm{d}z'\int \mathrm{d}^2\kappa\,\Psi(\kappa)[\exp(i\boldsymbol{\kappa\rho})\exp(-\kappa^2\hat{L}^2) - 1], \quad (24)$$

where

$$\boldsymbol{\rho} = \left[\mathbf{p}\sinh\left(\eta\frac{z'}{\mathcal{Z}}\right) + \boldsymbol{\xi}\sinh\left(\eta\frac{\mathcal{Z} - z'}{\mathcal{Z}}\right)\right]\Big/\sinh\eta,$$

$$\hat{L}^2 = \left(\frac{\Delta\mathcal{Z}}{4k_1k_2}\right)\left[\cosh\eta - \cosh\left(\eta\frac{\mathcal{Z} - 2z'}{\mathcal{Z}}\right)\right]\Big/\eta\sinh\eta.$$

Therefore

$$J = \frac{\pi}{2}k_1k_2\int_0^{\mathcal{Z}} \mathrm{d}z'\int \mathrm{d}^2\kappa\,\Psi(\kappa) \times [\exp(i\boldsymbol{\kappa\rho})\exp(-\kappa^2\hat{L}^2) - 1]. \quad (25)$$

We can see from Eq. (25) that $\hat{L}^2$ introduces an extra length scale into the problem, along with the Gaussian and the power-law (Kolmogorov) cutoffs. We will discuss its physical significance in Section 5.

Relation (21) and Eq. (25) constitute the main result of this section.

5. VARIATIONAL SOLUTIONS FOR KOLMOGOROV TURBULENCE

A. Normalized Spherical Wave Mutual Coherence Function

In this subsection we shall use $\Psi(\kappa)$—the difference in dielectric permittivity fluctuation spectra between Kolmogorov ($p = 5/3$) and Gaussian ($p = 2$) turbulence—to find J and eventually $\Gamma_{5/3}^N$.

In the case of the Kolmogorov spectrum, $\Phi_{5/3}^{\epsilon}(\kappa)$ is known to be[6,28]

$$\Phi_{5/3}^{\epsilon}(\kappa) = 0.033C_{\epsilon}^2\kappa^{-11/3}\exp(-\kappa^2/\kappa_m^2), \quad (26)$$

where C_ϵ^2 is the dielectric permittivity structure constant and $\kappa_m = 5.92/l_0$. Equation (26) corresponds to $A_{5/3} = -1.46C_\epsilon^2$ $(l_0 \ll \rho \ll L_0)$. Here l_0 is the inner scale size of turbulence, and L_0 is the outer scale size of turbulence. This approximation can be extended to all values of ρ in the following range of propagation distances[28]:

$$10.26C_\epsilon^{-2}k_0^{-2}L_0^{-5/3} \ll \mathcal{Z} \ll 10.26C_\epsilon^{-2}k_0^{-2}l_0^{-5/3}.$$

In the case of the Gaussian spectrum,[4]

$$\Phi_2^\epsilon(\kappa) = \frac{\langle\epsilon_1^2\rangle l^3}{8\pi\sqrt{\pi}}\exp\left(-\frac{1}{4}\kappa^2 l^2\right), \tag{27}$$

where $\langle\epsilon_1^2\rangle$ is the variance and l is the characteristic scale size of the Gaussian fluctuations. We can assume that $\langle\epsilon_1^2\rangle$ is related to the structure constant via[28]

$$\langle\epsilon_1^2\rangle = \frac{\tilde{C}C_\epsilon^2 L_0^{2/3}}{\sqrt{\pi}}.$$

The outer scale size of turbulence is introduced here to make $\tilde{C}$ dimensionless in a way that formally resembles the Kolmogorov dielectric permittivity variance; the variational approach is independent of the actual value of L_0 unless it is introduced into the Kolmogorov spectrum.

Equation (27) corresponds to $A_2 = -\tilde{C}C_\epsilon^2 L_0^{2/3}/l$ $(\rho \ll l)$. We will use $\tilde{C}$ and l as our variational parameters, tuning them to maximize the right-hand side of relation (21).

The difference of Eqs. (26) and (27) gives $\Psi(\kappa)$, which we now substitute into Eq. (25):

$$\begin{aligned} J &= \pi^2 k_1 k_2 \int_0^{\mathcal{Z}} \mathrm{d}z' \int_0^\infty \kappa \mathrm{d}\kappa [\Phi_{5/3}^\epsilon(\kappa) - \Phi_2^\epsilon(\kappa)] \\ &\quad \times [\exp(-\kappa^2\hat{L}^2)J_0(\kappa\rho) - 1] \\ &\equiv \pi^2 k_1 k_2 C_\epsilon^2 \int_0^{\mathcal{Z}} \mathrm{d}z' \{[I_{5/3}(\hat{L}^2, \rho) - I_{5/3}(0, 0)] \\ &\quad - [I_2(\hat{L}^2, \rho) - I_2(0, 0)]\}, \end{aligned} \tag{28}$$

where

$$I_{5/3} = 0.033\int_0^\infty \kappa \mathrm{d}\kappa \kappa^{-11/3} \exp[-\kappa^2(1/\kappa_m^2 + \hat{L}^2)]J_0(\kappa\rho),$$

$$I_2 = \frac{\tilde{C}L_0^{2/3}l^3}{8\pi^2}\int_0^\infty \kappa \mathrm{d}\kappa \exp[-\kappa^2(l^2/4 + \hat{L}^2)]J_0(\kappa\rho).$$

$J_0(z)$ is the zeroth-order Bessel function.

Taking the integrals above, we obtain for Eq. (28)

$$\begin{aligned} J &= \pi^2 k_1 k_2 C_\epsilon^2 \int_0^{\mathcal{Z}} \mathrm{d}z' \Bigg\{ \frac{0.033\Gamma(-5/6)}{2} \\ &\quad \times [\tilde{\kappa}_m^{-5/3}\ {}_1F_1(-5/6, 1; -\tilde{\kappa}_m^2\rho^2/4) - \kappa_m^{-5/3}] \\ &\quad - \frac{\tilde{C}L_0^{2/3}l}{4\pi^2}\left[\frac{l^2}{l^2 + 4\hat{L}^2}\left(1 - \frac{\rho^2}{l^2 + 4\hat{L}^2}\right) - 1\right]\Bigg\}, \end{aligned} \tag{29}$$

where $l_0 \ll \rho \ll l$, L_0 and $\tilde{\kappa}_m^2 = \kappa_m^2/(1 + \hat{L}^2\kappa_m^2)$. Here $\Gamma(z)$ is the gamma function and ${}_1F_1(a, c; z)$ is the confluent hypergeometric function of the first kind.

Finally, we are in a position to write down the normalized spherical wave MCF for Kolmogorov turbulence:

$$\begin{aligned} \Gamma_p^N &\geq \frac{\eta}{\sinh \eta}\exp(J)\exp\left[-\frac{1}{8}\Delta^2\mathcal{Z}A(0)\right]\exp\left[\frac{k_1k_2}{2\Delta\mathcal{Z}}(\boldsymbol{\xi} - \mathbf{p})^2\right] \\ &\quad \times \exp\left\{\left(-\frac{k_1k_2}{2\Delta\mathcal{Z}}\right)\frac{\eta}{\sinh\eta}[(\boldsymbol{\xi}^2 + \mathbf{p}^2)\cosh\eta - 2\boldsymbol{\xi}\mathbf{p}]\right\}, \end{aligned} \tag{30}$$

where the expression for J was obtained in Eq. (29). We recall that[28]

$$A(0) = 0.7816C_\epsilon^2 L_0^{5/3}.$$

The right-hand side of relation (30) should be maximized with respect to $\tilde{C}$ and l to obtain the best approximation for Γ_p^N. However, inspection of Eq. (29) and relation (30) shows that they in fact depend only on $\tilde{C}/l$, once $l \gg \rho$, $\hat{L}$ for all values of z'. We can then fix l at some large constant value and vary only $\tilde{C}$ whenever numerical maximization is called for below.

Finally, we obtain the $\Delta = 0$ limiting form of Γ_p^N and show that it reduces to an exact expression in this case. First, it is easy to show that

$$\begin{aligned} \lim_{\Delta\to 0}\Gamma_2^N &= \exp\left(\frac{1}{4}k_0^2A_2\mathcal{Z}\int_0^1 \mathrm{d}t|\boldsymbol{\rho}'|^2\right) \\ &= \exp\left[\frac{1}{12}k_0^2A_2\mathcal{Z}(\boldsymbol{\xi}^2 + \mathbf{p}^2 + \boldsymbol{\xi}\mathbf{p})\right], \end{aligned} \tag{31}$$

where

$$\boldsymbol{\rho}' = t\mathbf{p} + (1 - t)\boldsymbol{\xi}.$$

Second,

$$\begin{aligned} \lim_{\Delta\to 0} J &= \frac{\pi}{2}k_0^2\mathcal{Z}\int_0^1 \mathrm{d}t \int \mathrm{d}^2\kappa\Psi(\kappa)[\exp(i\boldsymbol{\kappa}\boldsymbol{\rho}') - 1] \\ &= \frac{1}{4}k_0^2\mathcal{Z}\int_0^1 \mathrm{d}t[A_{5/3}|\boldsymbol{\rho}'|^{5/3} - A_2|\boldsymbol{\rho}'|^2]. \end{aligned} \tag{32}$$

Combining Eqs. (31) and (32) gives us

$$\Gamma_{5/3}^N = \exp\left(\frac{1}{4}k_0^2A_{5/3}\mathcal{Z}\int_0^1 \mathrm{d}t|\boldsymbol{\rho}'|^{5/3}\right), \tag{33}$$

which is a correct form of the MCF when $\Delta = 0$.[24,28] One can easily obtain, for example, a well-known expression for the plane-wave MCF by performing straightforward integrations over the initial source distribution:

$$\Gamma_{5/3}^{\mathrm{pl}}(\mathbf{p}) = \Gamma_{5/3}^N(\mathbf{p}, \mathbf{p}) = \exp(\tfrac{1}{4}k_0^2A_{5/3}\mathcal{Z}|\mathbf{p}|^{5/3}). \tag{34}$$

In the following subsections, we will find $\Gamma_{5/3}^N$ by carrying out its maximization with respect to $\tilde{C}$ and l numerically; we will also reverse the analytical continuation used in the formulation of the variational principle and integrate when necessary over initial source distributions to obtain spherical wave, plane-wave, and Gaussian beam two-frequency MCFs in Kolmogorov turbulence.

In the interests of clarity, we shall omit the $\exp\{-[\Delta^2 \mathcal{Z} A(0)]/8\}$ prefactor from all further results, unless explicitly indicated otherwise.

B. Spherical Wave Mutual Coherence Function

We start by exploring some special cases related to the two-frequency MCF of two point sources located in the $z = 0$ plane. The spherical MCF will serve as a basis for the other two, which can be derived by integrating over initial point-source distributions.

We provide an illustration of the variational technique by choosing both the source and the observation points to be at the origin: $\mathbf{R} = \vec{0}$, $\mathbf{R}^0 = \vec{0}$, $\mathbf{p} = \vec{0}$, and $\boldsymbol{\xi} = \vec{0}$. This will prove to be a simple example to consider; the method, however, is in no way limited to this special case.

We obtain, after analytical continuation back into complex space,

$$\Gamma_{5/3}^{\text{sph}}(\vec{0};\, k_0,\, \Delta) = \frac{k_1 k_2}{(2\pi \mathcal{Z})^2} \frac{\eta}{\sin \eta} \exp\left(k_1 k_2 C_\epsilon^2 \int_0^{\mathcal{Z}} \mathrm{d}z' \left\{ \tilde{C} L_0^{2/3} l \frac{\hat{L}^2}{l^2 + 4\hat{L}^2} - 0.11\pi^2 \kappa_m^{-5/3}[(1 + \hat{L}^2 \kappa_m^2)^{5/6} - 1] \right\}\right), \tag{35}$$

where

$$\hat{L}^2 = \frac{i}{4} \frac{\Delta \mathcal{Z}}{k_1 k_2} \frac{\cos \eta - \cos\left(\eta \dfrac{\mathcal{Z} - 2z'}{\mathcal{Z}}\right)}{\eta \sin \eta}.$$

Comparison of $\Gamma_{5/3}^{N}(\tilde{C} = 0)$ with $\Gamma_{5/3}^{N}(\tilde{C} = \tilde{C}_{\max})$ for a wide range of input parameters leads us to believe that the former is sufficient when only qualitative estimates are required; Eq. (35) then simplifies further to become

$$\Gamma_{5/3,0}^{\text{sph}}(\vec{0};\, k_0, \Delta) = \frac{k_1 k_2}{(2\pi \mathcal{Z})^2} \exp\left\{ -0.11\pi^2 k_1 k_2 C_\epsilon^2 \times \int_0^{\mathcal{Z}} \mathrm{d}z' \kappa_m^{-5/3}[(1 + \hat{L}^2 \kappa_m^2)^{5/6} - 1] \right\}, \tag{36}$$

where

$$\hat{L}^2(t) = -\frac{i \Delta \mathcal{Z}}{2 k_1 k_2} t(1 - t).$$

We see that when $\mathrm{Re,Im}(\hat{L}^2 \kappa_m^2) \gg 1$ for any t (regions for which $\mathrm{Re,Im}(\hat{L}^2 \kappa_m^2) \ll 1$ do not contribute to the integral), we obtain qualitatively

$$\Gamma_{5/3,0}^{\text{sph}} \sim \exp[-(\Delta/k_{5/3,\text{coh}})^{5/6}], \tag{37}$$

where $k_{5/3,\text{coh}} = 6.703 C_\epsilon^{-12/5} \mathcal{Z}^{-11/5} k_0^{-2/5}$ is the plane-wave coherence bandwidth wave number.[18,22,28]

In Fig. 1 we plot the magnitude and phase of $\Gamma_{5/3}^{\text{sph}}(\vec{0};\, k_0,\, \Delta)/\Gamma_{5/3}^{\text{sph}}(\vec{0};\, k_0,\, \Delta = 0)$ as a function of the difference wave number normalized by the coherence bandwidth[18,28]:

$$\Delta_n \equiv (\Delta/k_{5/3,\text{coh}})^{5/11}.$$

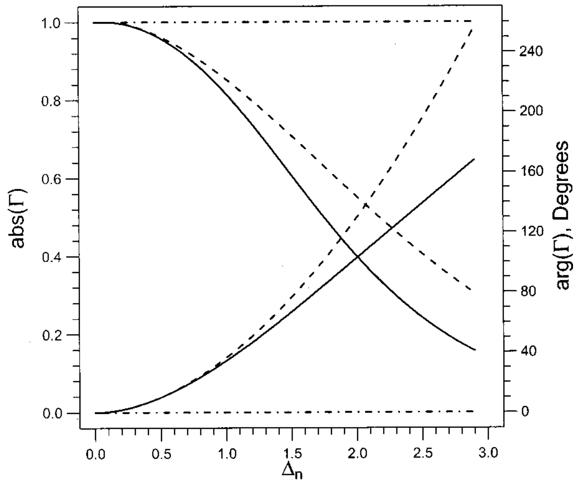


Fig. 1. $\lambda_0 = 0.6943\ \mu\text{m}$, $\mathcal{Z} = 10\ \text{km}$, and $C_\epsilon^2 = 4.856 \times 10^{-14}\ \text{m}^{-2/3}$. Absolute magnitude and phase of $\Gamma_{5/3}^{\text{sph}}(R = R^0 = p = \xi = 0;\, k_0,\, \Delta)/\Gamma_{5/3}^{\text{sph}}(R = R^0 = p = \xi = 0;\, k_0,\, \Delta = 0)$ versus Δ_n. Solid curves, $\tilde{C} = \tilde{C}_{\max}$; dashed curves, $\tilde{C} = 0$; dashed–dotted curves, HF approximation.

We show the results given by Eq. (35), its approximation [Eq. (36)] with $\tilde{C} = 0$, and the extended HF principle.[23] The full variational result corresponds to the point-source two-frequency Kolmogorov MCF obtained earlier in Ref. 20 via modal expansion techniques, whereas an approximate expression [Eq. (36)] provides a qualitatively useful estimate superior to the HF expression derived in Ref. 23, which fails to capture any variation in this limit.

At this point, we elucidate further the physical meaning of $\hat{L}(z')$, first introduced in Eq. (24). Consider an average value of $\hat{L}^2$ over the propagation path:

$$\langle \hat{L}^2 \rangle \equiv \int_0^1 \mathrm{d}t \hat{L}^2(t) = \frac{i}{4} \frac{\Delta \mathcal{Z}}{k_1 k_2} \frac{\eta \cos \eta - \sin \eta}{\eta^2 \sin \eta}. \tag{38}$$

We expect η to be of $\mathcal{O}(1)$ in most cases, and thus essentially

$$\langle \hat{L}^2 \rangle \sim \frac{\Delta \mathcal{Z}}{k_1 k_2} \sim \frac{\Delta}{k_0} \mathcal{Z} \lambda_0,$$

where k_0 is the central wave number and $\lambda_0 = 2\pi/k_0$ is the central wavelength. We recognize $\sqrt{\mathcal{Z}\lambda_0}$ as the measure of the extent of diffractional spreading at the screen located at $\mathcal{Z}$; it is well known[6] that $\sqrt{\mathcal{Z}\lambda_0} \ll l_0$ has to hold for geometrical optics to be valid; this leads to $\langle \hat{L}^2 \rangle \kappa_m^2 \ll 1$ (since $\Delta/k_0 \leq 1$). Consequently, Eq. (29) simplifies significantly because $\tilde{\kappa}_m^2 \to \kappa_m^2$ in this case.

However, in many problems of practical interest diffractional effects are important, which means that unless Δ/k_0 is extremely small, $\tilde{\kappa}_m^2 \to \hat{L}^{-2}$. Thus in a number of cases the following hierarchy of relevant scales emerges:

$$l_0 \ll \mathrm{Re,Im}\langle \hat{L} \rangle \ll |\mathbf{p}|,\ |\boldsymbol{\xi}| \ll l, L_0. \tag{39}$$

The middle inequality allows us to replace ${}_1F_1$ in Eq. (29) by its asymptotic expansion, valid for $|z| \gg 1$[29]:

$$ {}_1F_1(-5/6, 1; z) \simeq \frac{(-z)^{5/6}}{\Gamma(11/6)} + \frac{\exp(z) z^{-11/6}}{\Gamma(-5/6)}. \tag{40} $$

Moreover, we can omit the second term on the right-hand side of expression (40) by the following argument: When $\tilde{C} = 0$, $-\pi \leqslant -\tilde{\kappa}_m^2 \rho^2/4 \leqslant -\pi/2$, and the exponential term is negligible. We can imagine increasing $\tilde{C}$ from 0 to $\tilde{C}_{\max}$ in some smooth way; the phase of the ${}_1F_1$ argument may then cross the negative imaginary axis, making the real part of the argument positive. This will happen first in the vicinity of $t = 1/2$ (the middle of the propagation path). However, even in this case the rapid oscillations produced by the exponential term in expression (40) will average out to zero in the process of integration over t. This argument breaks down if the phase crosses the positive real axis as $\tilde{C}$ increases; however, this would lead to huge nonphysical contributions to the integral and is consequently never chosen by the variational procedure.

Keeping the above argument in mind and utilizing inequalities (39), we can considerably simplify Eq. (29) for J:

$$ J = -0.365 k_1 k_2 C_\epsilon^2 \mathcal{Z} \int_0^1 \mathrm{d}t |\boldsymbol{\rho}|^{5/3} + \frac{k_1 k_2 C_\epsilon^2 \tilde{C} L_0^{2/3} \mathcal{Z}}{4l} \int_0^1 \mathrm{d}t |\boldsymbol{\rho}|^2, \tag{41} $$

where $\boldsymbol{\rho}$ is the complex version of the definition introduced below Eq. (24).

We see that in this case, the $\tilde{C} = 0$ limit correctly reproduces the HF expression[23]

$$ J = -0.365 k_1 k_2 C_\epsilon^2 \mathcal{Z} \int_0^1 \mathrm{d}t |\boldsymbol{\rho}'|^{5/3}, \tag{42} $$

whereas Eq. (41) provides a systematic improvement over the HF limit.

C. Plane-Wave Mutual Coherence Function

After integrating over the uniform initial source distribution, we obtain the following result for a unit-amplitude plane-wave two-frequency MCF:

$$ \Gamma_{5/3}^{\mathrm{pl}}(\mathbf{p}; k_0, \Delta) = \frac{i k_1 k_2}{2\pi \mathcal{Z} \Delta} \int \mathrm{d}^2 \xi \Gamma_{5/3}^N(\mathbf{p}, \boldsymbol{\xi}; k_0, \Delta) \times \exp\left[-\frac{i k_1 k_2}{2 \Delta \mathcal{Z}} (\mathbf{p} - \boldsymbol{\xi})^2 \right]. \tag{43} $$

For computational efficiency, we use a simplified Eq. (41) to compute $\Gamma_{5/3}^N$ with arbitrary $\mathbf{p}$, $\boldsymbol{\xi}$. As the results below clearly indicate, this simplification does not introduce a significant error, except in the asymptotic regime.

In Fig. 2 we show the magnitude and phase of $\Gamma_{5/3}^{\mathrm{pl}}(\mathbf{p}; k_0, \Delta)$ as a function of Δ_n. To enable direct comparison with previous results,[3,18,22,28] we define

$$ \xi_n \equiv (-k_0^4 A_{5/3}/2\Delta)^{3/11} |\mathbf{p}| $$

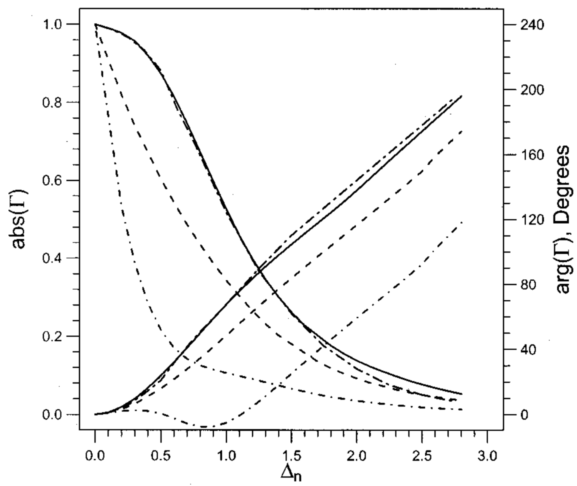


Fig. 2. Absolute magnitude and phase of $\Gamma_{5/3}^{\mathrm{pl}}(p; k_0, \Delta)$ versus Δ_n. Solid curves, $\xi_n = 0$; dashed curves, $\xi_n = 1$; dashed–dotted curves, $\xi_n = 2$; long-dashed–dotted curves, a $\xi_n = 0$ numerical solution from Ref. 28.

and plot $\Gamma_{5/3}^{\mathrm{pl}}$ with $\xi_n = 0, 1, 2$. We also plot $\xi_n = 0$ from Ref. 28 for comparison. As we can see, the agreement is quite good everywhere except in the asymptotic regime, in which it can be improved by use of the lowest-order-mode results from Ref. 22 or by use of the full Eq. (29) for J instead of the simplified one [Eq. (41)] in the integrand of Eq. (43).

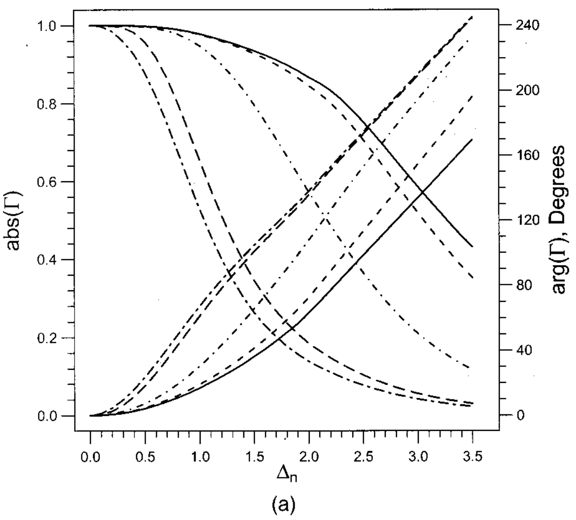
D. Gaussian Beam Mutual Coherence Function

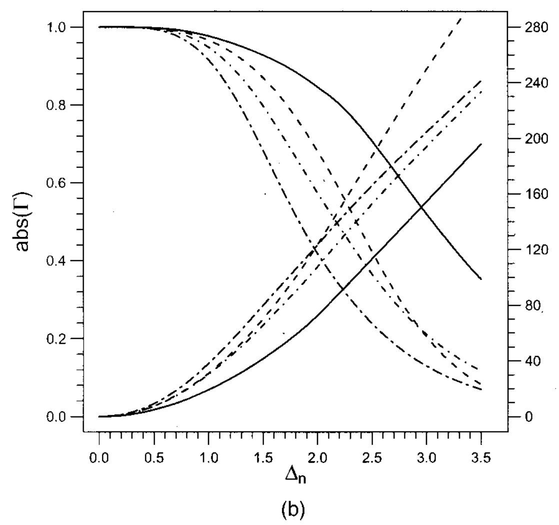
Computations for $\Gamma_{5/3}^{\mathrm{Gaus}}$ are no more difficult than those for $\Gamma_{5/3}^{\mathrm{pl}}$ within the current framework; in fact, we expect the two-dimensional initial source distribution integral to converge better than that for the plane-wave case. We obtain the following expression for the Gaussian beam two-frequency MCF:

$$ \begin{aligned} \Gamma_{5/3}^{\mathrm{Gaus}}(\mathbf{R}, \mathbf{p}; k_0, \Delta) &= \frac{k_1 k_2}{4\pi\alpha \mathcal{Z}^2} \exp\left(\frac{i\Delta}{2\mathcal{Z}} \mathbf{R}^2 \right) \int \mathrm{d}^2 \xi \Gamma_{5/3}^N(\mathbf{p}, \boldsymbol{\xi}; k_0, \Delta) \\ &\times \exp\left(-\frac{\gamma}{4} \boldsymbol{\xi}^2 \right) \exp\left[\frac{i\Delta}{8\mathcal{Z}} (\mathbf{p} - \boldsymbol{\xi})^2 \right] \\ &\times \exp\left[\frac{i k_0}{\mathcal{Z}} \mathbf{R}(\mathbf{p} - \boldsymbol{\xi}) \right] \\ &\times \exp\left[-\frac{1}{4\alpha} \left(\frac{\Delta}{\mathcal{Z}} \mathbf{R} + \frac{k_0}{\tilde{F}} \boldsymbol{\xi} + \frac{k_0}{\mathcal{Z}} \mathbf{p} \right)^2 \right], \end{aligned} \tag{44} $$

where α and $\tilde{F}$ were defined after Eq. (16) and $\gamma = 2/W_0^2 + i\Delta/2F_0$. The simplified Eq. (41) was used again for computations in this section.

In Fig. 3 we show the magnitude and phase of $\Gamma_{5/3}^{\mathrm{Gaus}}(\mathbf{R}, \mathbf{p}; k_0, \Delta)/\Gamma_{5/3}^{\mathrm{Gaus}}(\mathbf{R}, \mathbf{p}; k_0, \Delta = 0)$ as a function of Δ_n for a number of different input parameters. We choose $\lambda_0 = 0.6943\ \mu\mathrm{m}$, $\mathcal{Z} = 10\ \mathrm{km}$, and $C_\epsilon^2 = 4.856 \times 10^{-14}\ \mathrm{m}^{-2/3}$ for all numerical examples to follow.





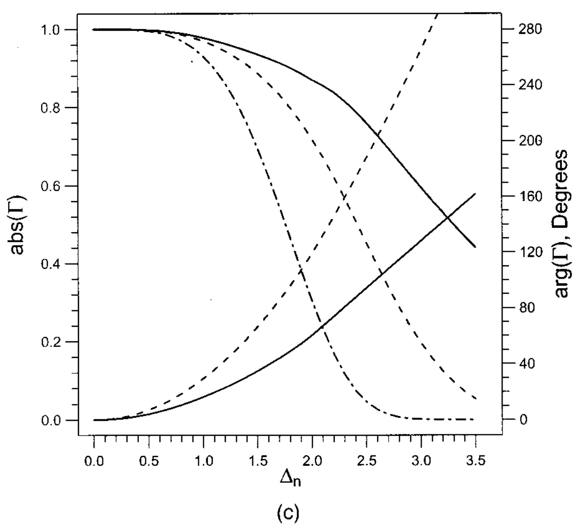


Fig. 3. $\lambda_0 = 0.6943\ \mu\text{m}$, $\mathcal{Z} = 10\ \text{km}$, and $C_\epsilon^2 = 4.856 \times 10^{-14}\ \text{m}^{-2/3}$. (a) Absolute magnitude and phase of $\Gamma_{5/3}^{\text{Gaus}}(R = p = 0;\ k_0,\ \Delta)/\Gamma_{5/3}^{\text{Gaus}}(R = p = 0;\ k_0,\ \Delta = 0)$ versus Δ_n for a collimated beam. Solid curves, $W_0 = 4$ cm; dashed curves, $W_0 = 10$ cm; dashed–dotted curves, $W_0 = 25$ cm; long-dashed curves, $W_0 = 1$ m; long-dashed–dotted curves, $W_0 \geqslant 10$ m. (b) Absolute magnitude and phase of $\Gamma_{5/3}^{\text{Gaus}}(R,\ p = 0;\ k_0,\ \Delta)/\Gamma_{5/3}^{\text{Gaus}}(R,\ p = 0;\ k_0,\ \Delta = 0)$ versus Δ_n for a collimated beam. $W_0 = 10$ cm: solid curves, $R = 0$ cm, dashed curves, $R = 15$ cm. $W_0 = 25$ cm: dashed–dotted curves, $R = 0$ cm; long-dashed–dotted curves, $R = 15$ cm. (c) Absolute magnitude and phase of $\Gamma_{5/3}^{\text{Gaus}}(R,\ p = 0;\ k_0,\ \Delta)/\Gamma_{5/3}^{\text{Gaus}}(R,\ p = 0;\ k_0,\ \Delta = 0)$ versus Δ_n for a beam focused at $\mathcal{Z}$. Solid curves, $W_0 = 10$ and 25 cm and $R = 0$ cm. Dashed phase curve, $W_0 = 10$ and 25 cm and $R = 15$ cm. Dashed absolute magnitude curve, $W_0 = 10$ cm and $R = 15$ cm. Long-dashed–dotted absolute magnitude curve, $W_0 = 25$ cm and $R = 15$ cm.

Figure 3(a) shows the results for a collimated beam with $|\mathbf{R}| = |\mathbf{p}| = 0$ and different aperture sizes W_0. We find that, unlike the Gaussian turbulence case,[4] the $W_0 = 1$-m curve is still substantially different from the plane-wave curve, especially in absolute magnitude; however, the $W_0 = 10$-m curve virtually coincides with it. In our calculations, the bandwidth is maximum when $W_0 = 4$ cm; its value is substantially higher than that for a point source.

In Fig. 3(b) we plot off-axis ($|\mathbf{R}| = 15$ cm) and on-axis ($|\mathbf{R}| = 0$ cm) curves for $W_0 = 10$ cm and $W_0 = 25$ cm. We observe that the bandwidth decreases as we move away from the beam axis. This effect is less pronounced for $W_0 = 25$ cm, since results for collimated beams should approach those for a plane wave with increasing aperture sizes.

Finally, the focused beam case is illustrated in Fig. 3(c). The $|\mathbf{R}| = |\mathbf{p}| = 0$ absolute magnitude curve, and both phase curves are the same for $W_0 = 10$ cm and $W_0 = 25$ cm. This also holds in the Gaussian turbulence case.[4] We can explain this effect by applying Eq. (44) to the focused beam:

$$\Gamma_{5/3}^{\text{Gaus},f}(\mathbf{R},\ \mathbf{p};\ k_0,\ \Delta) = \frac{k_1 k_2 W_0^2}{8\pi\mathcal{Z}^2}\exp\left[\frac{i\Delta}{2\mathcal{Z}}\left(\mathbf{R}^2 + \frac{\mathbf{p}^2}{4}\right)\right] \times \exp\left[-\frac{W_0^2}{8\mathcal{Z}^2}(\Delta\mathbf{R} + k_0\mathbf{p})^2\right]\int \mathrm{d}^2\xi\,\Gamma_{5/3}^{N}(\mathbf{p},\ \boldsymbol{\xi};\ k_0,\ \Delta) \times \exp\left(-\frac{\boldsymbol{\xi}^2}{2W_0^2}\right)\exp\left(-\frac{i\Delta}{4\mathcal{Z}}\boldsymbol{\xi}\mathbf{p}\right)\exp\left[\frac{ik_0}{\mathcal{Z}}\mathbf{R}(\mathbf{p} - \boldsymbol{\xi})\right]. \tag{45}$$

In most practical situations, $W_0 \gg l_{\text{cor}}$, where l_{cor} is the correlation length associated with $\Gamma_{5/3}^{N}$. But then the integral in the expression above is virtually independent of W_0. The only W_0 dependence left is in the second exponent on the right-hand side of Eq. (45). This exponent does not affect the phase and goes to 0 when $|\mathbf{R}| = |\mathbf{p}| = 0$.

6. CONCLUSIONS

In this paper we have used a path-integral variational technique to find the two-frequency mutual coherence function with nonquadratic turbulence structure functions. The method we developed is quite general with respect to the initial source distribution and the spectrum of refractive-index fluctuations. In particular, it may form a basis for the systematic study of the effects that inner and outer scale sizes and more realistic atmospheric turbulence models[30] have on pulse propagation in the atmosphere.

We would like to emphasize as well that the variational method presented in this paper is essentially analytical; even though one-dimensional numerical maximization was carried out to find $\tilde{C}_{\max}$, analytical approximations can be used as a rough guess of its value; even setting $\tilde{C} = 0$ produces results overall superior to the extended Huygens–Fresnel expression. However, in general, we still have to integrate numerically over the initial source distribution (this is true as well for the extended HF and other approximate expressions).

In this paper the variational method was applied to finding spherical wave, plane-wave, and Gaussian beam MCFs in Kolmogorov turbulence. Results obtained in such a way agree quite well with those previously published by other authors for plane and spherical waves. Moreover, the nature of the method allows us to extend it relatively easily to the Gaussian beam propagation problem. In this way, we obtain a number of results not previously available in the literature, with two observation points at arbitrary off-axis locations, separated by arbitrary distances (within the range of applicability of the Kolmogorov turbulence model).

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