Final Exam Solutions

(1.) The arrival prob. is

$$W(n+1|n,t) = \alpha$$

Then the master eq'n is

$$(*) \quad \frac{\partial P(n,t|0,0)}{\partial t} = \alpha\,[\underbrace{P(n-1,t|0,0)}_{\text{gain}} - \underbrace{P(n,t|0,0)}_{\text{loss}}]$$

The characteristic function is given by

$$G(s,t) = \sum_{n=0}^{\infty} P(n,t|0,0)\, e^{ins}$$

Apply $\sum_{n=0}^{\infty} e^{ins} \ldots$ to both sides of Eq. (*) :

$$\frac{\partial G(s,t)}{\partial t} = \alpha \underbrace{\sum_{n=1}^{\infty}}_{n=0 \text{ is } \emptyset} P(n-1,t|0,0)\, \underbrace{e^{ins}}_{n-1=m} - \alpha\, G(s,t) =$$

$$= \alpha \sum_{m=0}^{\infty} P(m,t|0,0)\, e^{i(m+1)s} - \alpha\, G(s,t) =$$

$$= \alpha\, G(s,t)\, [e^{is} - 1].$$

Then $G(s,t) = e^{\alpha t [e^{is}-1]}$ $\quad (**)$

Note that $P(n,0|0,0) = \delta_{n,0} \Rightarrow$

$\Rightarrow G(s,0) = 1$, as in $(**)$

So the initial condition is OK.

Finally, we need to get $P(n,t|0,0)$ from $G(s,t)$. Note that

$$\frac{1}{2\pi}\int_0^{2\pi} ds\, G(s,t)\, e^{-ins} = \frac{1}{2\pi}\int_0^{2\pi} ds \left(\sum_{m=0}^{\infty} P(m,t|0,0)\, e^{ims}\right) \times$$

$$\times\, e^{-ins} = \sum_{m=0}^{\infty} P(m,t|0,0) \underbrace{\left[\frac{1}{2\pi}\int_0^{2\pi} ds\, e^{i(m-n)s}\right]}_{\delta_{m,n}} =$$

$$= P(n,t|0,0).$$

So, $P(n,t|0,0) = \frac{1}{2\pi}\int_0^{2\pi} ds\, e^{-ins}\, e^{\alpha t[e^{is}-1]} =$

$u = e^{is}$

$ds = \frac{du}{iu}$

$$= \frac{1}{2\pi i}\oint \frac{du}{u^{n+1}}\, e^{\alpha t[u-1]}.$$

circle around the origin

Recall that for $y(x) = \sum_n A_n x^n$,

$$A_n = \frac{1}{2\pi i}\oint \frac{y(x)}{x^{n+1}}\, dx$$

Lagrange inversion formula

So, $P(n,t|0,0) = e^{-\alpha t}\frac{(\alpha t)^n}{n!}$ $\Leftarrow$ Poisson distribution (as expected!), with $\underline{\lambda = \alpha t}$

2. a) The mean-field eq'n is simply

$\langle \dot{n} \rangle = \langle n \rangle$, or

$\langle n \rangle = e^t$ with $\langle n \rangle|_{t=0} = 1$.

The master eq'n is

$$\dot{P}_n(t) = (n-1) P_{n-1}(t) - n P_n(t) \quad [P_n(0) = \delta_{n,1}]$$

Use $P_n = A a^{n-1}$ as advised:

$$\sum_{n=1}^{\infty} P_n = 1 \implies A = 1 - a.$$

Then $-\dot{a} a^{n-1} + (1-a)(n-1) a^{n-2} \dot{a} =$

$$= (n-1) \underbrace{A}_{(1-a)} a^{n-2} - n A (1-a) a^{n-1}, \text{ or}$$

$$\dot{a} [\underbrace{(n-1)(1-a) - a}_{(n-1) - na}] = (n-1)(1-a) - n(1-a)a =$$

$$= [(n-1) - na](1-a),$$

$$\dot{a} = 1 - a \implies a = 1 - e^{-t},$$

$$P_n(t) = e^{-t}(1 - e^{-t})^{n-1}.$$

Note that $P_n(0) = \delta_{n,1}$ is respected

Next, $\langle n \rangle = \sum_{n=1}^{\infty} n P_n = e^t$, as in the mean-field eq'n

However, $\sigma^2 = e^{2t} - e^t \to e^{2t}$ as $t \uparrow$, so that $\sigma \sim \langle n \rangle$. Thus fluct's are large and they do not go away as $t \uparrow$.

b) Mean-field eq'n:

$$\langle \dot{n} \rangle = (\lambda-\mu)\langle n \rangle,$$
$$\langle n \rangle = e^{(\lambda-\mu)t} \quad [\langle n \rangle|_{t=0} = 1]$$

Master eq'n:

$$\dot{P}_n = \lambda(n-1)P_{n-1} - (\lambda+\mu)nP_n + \mu(n+1)P_{n+1}$$

$\underline{n \geq 1}$: $\Rightarrow$ use $P_n = A a^{n-1}$ as before, but note that $A \neq 1-a$ anymore since $P_0(t)$ can be $\neq 0$. Rather, we have two equations:

$$\dot{A}a^{n-1} + (n-1)Aa^{n-2}\dot{a} =$$
$$= \lambda(n-1)Aa^{n-2} - (\lambda+\mu)nAa^{n-1} + \mu(n+1)Aa^{n},$$
$$\dot{A}a + (n-1)A\dot{a} = \lambda(n-1)A - (\lambda+\mu)nAa + \mu(n+1)Aa^2,$$
$$\dot{A}a - A\dot{a} + nA\dot{a} = -\lambda A + \mu Aa^2 + n[\lambda A - (\lambda+\mu)Aa + \mu Aa^2],$$ yielding

(1) $\dot{A}a - A\dot{a} = A(\mu a^2 - \lambda)$ $\leftarrow$ "n-indep." eq'n

(2) $\dot{a} = \lambda - (\lambda+\mu)a + \mu a^2$ $\leftarrow$ "n-dep." eq'n

These two eq's need to be satisfied separately.

Solve (2) first (in Mathematica):

$$a(t) = \frac{e^{(\lambda-\mu)t} - 1}{e^{(\lambda-\mu)t} - \mu/\lambda} \Leftarrow a(0) = 0, \text{ as dictated by the initial condition}$$

Likewise, $A(t) = e^{(\lambda-\mu)t}\left(\frac{1-\mu/\lambda}{e^{(\lambda-\mu)t}-\mu/\lambda}\right)^2$.

This gives $P_n(t)$.

The survival prob.

$$S(t) = 1 - P_0(t) = \sum_{n=1}^{\infty} P_n(t) =$$

$$= e^{(\lambda-\mu)t}\left(\frac{1-\mu/\lambda}{e^{(\lambda-\mu)t}-\mu/\lambda}\right)^2 \underbrace{\sum_{n=1}^{\infty} a^n}_{\frac{1}{1-a}} =$$

$$= e^{(\lambda-\mu)t}\left(\frac{1-\mu/\lambda}{e^{(\lambda-\mu)t}-\mu/\lambda}\right)^2 \frac{e^{(\lambda-\mu)t}-\mu/\lambda}{1-\mu/\lambda} =$$

$$= e^{(\lambda-\mu)t}\,\frac{1-\mu/\lambda}{e^{(\lambda-\mu)t}-\mu/\lambda}.$$

$t\to\infty$: $\mu<\lambda$ $\quad S(t) \to 1-\frac{\mu}{\lambda}$,

$\mu>\lambda$ $\quad S(t) \to 0$,

$\mu=\lambda$ $\quad S(t) = 0$.

So, if death rate $\geq$ birth rate, the extinction is certain to occur.

Even if $\mu<\lambda$, $S(t)<1$, and the population may become extinct.

Finally,

$$\langle n\rangle = \sum_{n=1}^{\infty} n P_n = A \sum_{n=1}^{\infty} n a^{n-1} =$$

$$= \frac{A}{(1-a)^2} = e^{(\lambda-\mu)t}$$ ← consistent with mean-field result

$$A = e^{(\lambda-\mu)t}(1-a)^2$$

$$\sigma^2 = \langle n^2\rangle - \langle n\rangle^2 = \frac{A(1+a)}{(1-a)^3} - \frac{A^2}{(1-a)^4} =$$

$$= \frac{e^{(\lambda-\mu)t}(1+a)}{1-a} - e^{2(\lambda-\mu)t} =$$

$$= \frac{e^{(\lambda-\mu)t}(1+a) - e^{2(\lambda-\mu)t}(1-a)}{1-a} =$$

$$= \frac{e^{(\lambda-\mu)t}\left[e^{(\lambda-\mu)t} - 1\right](\mu+\lambda)}{\lambda-\mu}.$$

$\lambda > \mu$: $\sigma^2 \to \frac{\lambda+\mu}{\lambda-\mu} e^{2(\lambda-\mu)t}$ as $t \uparrow$, becomes very large

$\lambda < \mu$: $\sigma^2 \to 0$ [but extinction is certain anyway]

$\lambda = \mu$: $\sigma^2 \to \frac{(\lambda-\mu)t}{\lambda-\mu}(\mu+\lambda) = 2\lambda t.$

$\sigma^2 \sim t$ & $\langle n\rangle = \text{const}$ in this regime.

3. Consider maximizing

$$\frac{S}{k_B} = -\sum_i p_i \log p_i$$ under two

constraints:

1) $\sum_i p_i = 1$ (normalization)

2) $p_i = 0$ for microstates with $E_i \neq E$, where E is the total energy of the system:

$$\sum_i p_i (1 - \delta_{E,E_i}) = 0$$

$$\delta \left\{ \frac{S}{k_B} - \lambda \left[\sum_i p_i (1 - \delta_{E,E_i}) \right] - \nu \left[\sum_i p_i - 1 \right] \right\} = 0, \text{ or}$$

$\underline{E = E_j}$: $$\frac{\partial}{\partial p_j} \left[-\sum_i p_i \log p_i - \nu \left(\sum_i p_i - 1 \right) \right] =$$

$$= -\log p_j - 1 - \nu = 0,$$

$$\underline{\underline{p_j = e^{-\nu - 1}}}$$

$\underline{E \neq E_j}$: $$\frac{\partial}{\partial p_j} \left[-\sum_i p_i \log p_i - \nu \left(\sum_i p_i - 1 \right) - \lambda \sum_i p_i \right] =$$

$$= -\log p_j - 1 - \nu - \lambda = 0,$$

$$\underline{\underline{p_j = e^{-\nu - 1 - \lambda}}}$$

$p_j = 0$ if $E \neq E_j \Rightarrow \lambda = +\infty$

Then $\sum_{j, E = E_j} p_j = 1 \Rightarrow p_j = \frac{1}{\Omega(E)}$, where
if $E = E_j$

$\Omega(E)$ is the total # of microstates with energy E.

So, $$p_j = \frac{\delta_{E,E_j}}{\Omega(E)}.$$

[This is the microcanonical distribution]