Midterm

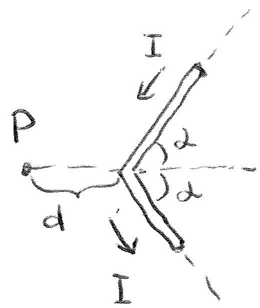
E&M 504

Spring 2017

Name:

Problem 1 [10 points]

Consider a bent wire shown below. If current I runs through the wire as indicated, what is the magnetic field $\vec{B}(\vec{r})$ at point P?



Note 1: the wire is infinite

Note 2: point P is on the axis of symmetry

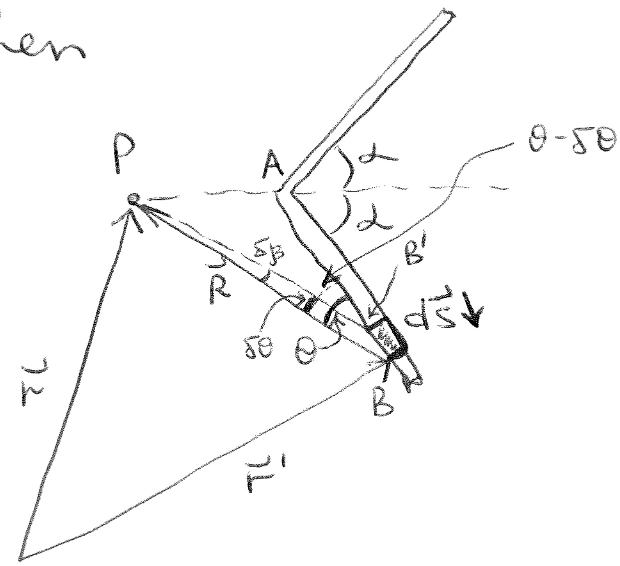
Problem 1 solution

Use Biot-Savart law:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{s} \times \vec{R}}{R^3},$$

where $\vec{R} = \vec{r} - \vec{r}\,'$

Then



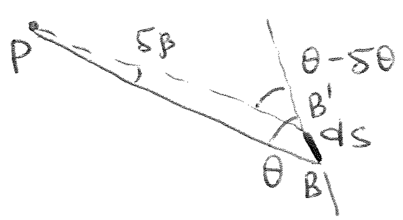
Then $\frac{d\vec{s} \times \vec{R}}{R^3} = \frac{ds}{R^2} \sin(\pi - \theta) = \frac{ds}{R^2} \sin\theta$

Law of sines, triangle PAB:

$$\frac{R}{\sin(\pi - \alpha)} = \frac{d}{\sin\theta}, \quad \text{or}$$

$$R = d\,\frac{\sin\alpha}{\sin\theta}$$

Law of sines, triangle PB'B:



$$(\pi - \theta + \delta\theta) + \theta + \delta\beta = \pi,$$

$$\delta\beta = \delta\theta$$

$R - \delta R \approx R$

$$\frac{ds}{\sin(\delta\beta)} \approx \frac{ds}{\delta\theta} \approx \frac{R}{\sin\theta}\,.$$

Finally,

$$\frac{ds}{R^2}\sin\theta = \frac{d\theta}{R} = \frac{\sin\theta\, d\theta}{d\sin\alpha}$$

This gives

$$|\vec{B}| = \underset{\uparrow}{2} \times \frac{\mu_0 I}{4\pi}\,\frac{1}{d\sin\alpha}\int_0^{\alpha} d\theta\,\sin\theta = \frac{\mu_0 I}{2\pi d}\,\frac{1-\cos\alpha}{\sin\alpha}.$$

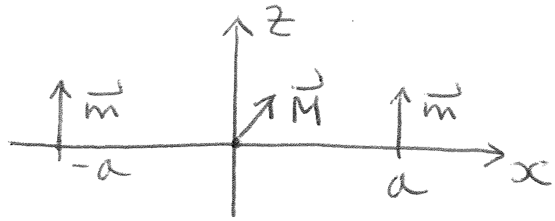
two pieces of wire

The direction of $\vec{B}$ is into the page.

...oblem 2 [10 points]

Consider two identical fixed dipoles located at $(\pm a, 0, 0)$: $\vec{m} = m\hat{z}$.
A third dipole, located at the origin, is free to rotate. If the dipole moment of the third dipole is $\vec{M}$, what is its orientation that minimizes the potential energy of the system?
What is the corresponding value of the <u>total</u> potential energy?

Problem 2 solution



Recall that the field due to the point dipole is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\vec{r})\,\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right]$$

(the dipole is at the origin)

Then $$\vec{B}(0,0,0) = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\vec{r}_1)\,\vec{r}_1}{r_1^5} - \frac{\vec{m}}{r_1^3}\right] +$$

$$+ \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\vec{r}_2)\,\vec{r}_2}{r_2^5} - \frac{\vec{m}}{r_2^3}\right] =$$

$$\begin{cases} \vec{r}_1 = -a\hat{x}, \\ \vec{r}_2 = a\hat{x} \end{cases} \qquad \begin{cases} \vec{m} \perp \vec{r}_1, \\ \vec{m} \perp \vec{r}_2 \end{cases}$$

$$= -\frac{\mu_0}{2\pi}\frac{\vec{m}}{a^3}$$

The potential energy of dipole $\vec{M}$ is given by $\hat{V}_B = -\vec{M}\cdot\vec{B}(0,0,0)$.

It is minimized when $\vec{M} \uparrow\uparrow \vec{B}$

and thus $\underline{\underline{\vec{M} \uparrow\downarrow \vec{m}}}$

$$\left[\quad \hat{V}_B = -\frac{\mu_0}{2\pi} \frac{mM}{d^3} \quad \right]$$

The <u>total</u> potential energy also includes the fixed potential energy between the $\pm a$ dipoles:

$$\hat{V}_{tot} = \hat{V}_B + \hat{V}_{12}, \quad \text{where}$$

$$\left[\quad \hat{V}_{12} = \frac{\mu_0}{4\pi} \frac{m^2}{8d^3} \quad \right]$$

Problem 3 [10 points]

A particle of charge q and mass m is subjected to the constant magnetic field $\vec{B} = B\hat{z}$.
At $t=0$, the particle is at the origin and its velocity is (v_{x0}, v_{y0}, v_{z0}).
What is the subsequent trajectory of the particle? Solve EoM, draw the trajectory and describe it.
Note: the particle is non-relativistic.

Problem 3 solution

The EoM is $\dot{\vec{p}} = q\,\vec{v}\times\vec{B}$, or

$$\dot{\vec{v}} = \frac{q}{m}\,\vec{v}\times\vec{B}$$

By components, $\omega = \frac{qB}{m}$ - the cyclotron frequency

$$\begin{cases} \dot{v}_x = \frac{q}{m} v_y B = \omega v_y, \\ \dot{v}_y = -\frac{q}{m} v_x B = -\omega v_x, \\ \dot{v}_z = 0 \end{cases} \qquad (*)$$

First, $v_z = \dot{z} = \text{const} = v_{z0} \Rightarrow z = v_{z0} t$ $(z(0)=0)$

Second, observe that

$$v_x \dot{v}_x + v_y \dot{v}_y = 0 \Rightarrow \frac{d}{dt}(\underbrace{v_x^2 + v_y^2}_{\equiv\, v_t^2}) = 0.$$

v_t^2, v_t is the transverse velocity

Then $v_t^2 = v_{t0}^2 = v_{x0}^2 + v_{y0}^2$, and v_t is conserved. Indeed, since Lorentz force does no work, the total kinetic energy stays constant. So, v_x & v_y can only rotate in a circle.

It is easy to see that (*) is solved by

$$\begin{cases} v_x = v_{t0}\cos(\omega t+\varphi), \\ v_y = -v_{t0}\sin(\omega t+\varphi). \end{cases}$$

Note that $\begin{cases} v_{x_0} = v_{t_0}\cos\varphi, \\ v_{y_0} = -v_{t_0}\sin\varphi \end{cases}$

Finally, $\begin{cases} x = x_0 + R\sin(\omega t+\varphi), \\ y = y_0 + R\cos(\omega t+\varphi). \end{cases}$

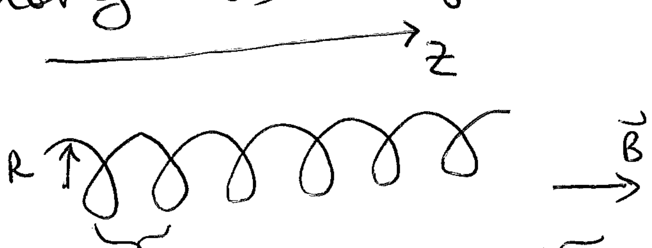
Here, $R = \dfrac{v_{t0}}{\omega}$.

$x(0)=0 \Rightarrow x_0 = -R\sin\varphi$

$y(0)=0 \Rightarrow y_0 = -R\cos\varphi$

So, $\begin{cases} x(t) = R\sin(\omega t+\varphi) - R\sin\varphi, \\ y(t) = R\cos(\omega t+\varphi) - R\cos\varphi, \\ z_{(t)} = v_{z_0} t. \end{cases}$

The trajectory is a spiral of radius R:



pitch $s = v_{z0}\dfrac{2\pi}{\omega}$