(11.11) $A_1 = e^{L}, \quad A_2 = e^{L+\delta L}$

Consider $\begin{cases} A_1(\lambda) = e^{\lambda L}, \\ A_2(\lambda) = e^{\lambda(L+\delta L)} \end{cases}$.

Now, $A(\lambda) = A_2(\lambda) A_1^{-1}(\lambda) \; {}_{*}\!= e^{\lambda(L+\delta L)} e^{-\lambda L}$.

Consider the Taylor series of $A(\lambda)$:

$$A(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} A^{(n)}(0)$$

$\underline{n=0}$: $A^{(0)}(0) = 1$

$\underline{n=1}$: $\lambda A^{(1)}(0) = \lambda(L+\delta L) - \lambda L = \lambda \delta L \Rightarrow$

$$\Rightarrow A^{(1)}(0) = \underline{\underline{\delta L}}.$$

Proceed by induction:

assume $A^{(n)}(0) = [L, A^{(n-1)}(0)]$.

Then $\frac{d}{d\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} A^{(n)}(0) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \underbrace{A^{(n)}(0)}_{[L, A^{(n-1)}(0)]} =$

$\uparrow$ $n=0$ term is indep. of λ

$$\overset{m=n-1}{=} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} A^{(m+1)}(0) \Rightarrow A^{(m+1)}(0) = [L, A^{(m)}(0)]$$

Finally,

$$A(\lambda=1) = 1 + \delta L + \frac{1}{2!}[L, \delta L] + $$
$$+ \frac{1}{3!}[L,[L,\delta L]] + \dots \quad , \text{ as desired.}$$

12.1

a) $L = -\frac{m}{2} u_\alpha u^\alpha - \frac{q}{c} u_\alpha A^\alpha =$

$$= -\frac{m}{2} g_{\alpha\beta} u^\beta u^\alpha - \frac{q}{c} g_{\alpha\beta} u^\beta A^\alpha .$$

Then $\frac{\partial L}{\partial u^\gamma} = -\frac{m}{2} g_{\alpha\beta} [\delta^\beta{}_\gamma u^\alpha + u^\beta \delta^\alpha{}_\gamma] -$

$$- \frac{q}{c} g_{\alpha\beta} \delta^\beta{}_\gamma A^\alpha = -\frac{m}{2}[u_\gamma + u_\gamma] - \frac{q}{c} A_\gamma ,$$

$$\frac{\partial L}{\partial x^\gamma} = -\frac{q}{c} g_{\alpha\beta} u^\beta \partial_\gamma A^\alpha = -\frac{q}{c} u_\alpha \partial_\gamma A^\alpha$$

EL eq's: $\underset{\substack{\uparrow \\ \text{proper time}}}{\frac{d}{d\tau}} \frac{\partial L}{\partial u^\alpha} = \partial_\gamma L$, or

$$\frac{d}{d\tau}[m u_\gamma + \frac{q}{c} A_\gamma] = \frac{q}{c} u^\alpha \partial_\gamma A_\alpha .$$

$$m \frac{du_\gamma}{d\tau} = \frac{q}{c}\Big[u^\alpha \partial_\gamma A_\alpha - \underbrace{\frac{dA_\gamma}{d\tau}}_{\frac{dx^\alpha}{d\tau}\frac{\partial A_\gamma}{\partial x^\alpha} = u^\alpha \partial_\alpha A_\gamma}\Big],$$

or $$m \frac{du_\gamma}{d\tau} = \frac{q}{c} u^\alpha [\underbrace{\partial_\gamma A_\alpha - \partial_\alpha A_\gamma}_{F_{\gamma\alpha}}] =$$

$$= \underline{\underline{\frac{q}{c} F_{\gamma\alpha} u^\alpha}} .$$

Covariant Lorentz force eq'n

b) $$p^\alpha = -\frac{\partial L}{\partial u_\alpha} = m u^\alpha + \frac{q}{c} A^\alpha$$

canonical momentum

$$H = p^\alpha u_\alpha + L = \frac{m}{2} \underbrace{u^\alpha u_\alpha}_{c^2} =$$

signs consistent with (12.34)

$$= \frac{1}{2m}\left(p^\alpha - \frac{q}{c} A^\alpha\right)\left(p_\alpha - \frac{q}{c} A_\alpha\right)$$

$$H = \frac{mc^2}{2} \leftarrow \text{Lorentz inv}$$

In space-time coordinates,

$$p^\alpha - \frac{q}{c} A^\alpha = \begin{pmatrix} p^0 - \frac{q}{c}\varphi \\ \vec{p} - \frac{q}{c}\vec{A} \end{pmatrix}.$$

Then

$$H = \frac{1}{2m}\left[(p^0)^2 - \vec{p}^{\,2} + \frac{q^2}{c^2}(\varphi^2 - \vec{A}^2) + \frac{2q}{c} \times \times (\vec{p}\cdot\vec{A} - p^0\varphi) \right].$$