HW #7

11.6 a) K - lab frame (ignore Earth motion)
K' - instantaneous rest frame of the ship

u - ship velocity in K
u' - ship velocity in K'
v - velocity of K' wrt K

Parallel velocities (ship is moving along the z-axis):

$$u = \frac{u'+v}{1+\frac{vu'}{c^2}} \quad (*)$$

inertial frames

Then $$\frac{du}{dt} = \frac{\left(1+\frac{u'v}{c^2}\right)\left(\frac{du'}{dt}+\cancel{\frac{dv}{dt}}\right) - \frac{u'+v}{c^2}\left(\frac{du'}{dt}v + \cancel{\frac{dv}{dt}}u'\right)}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$= \frac{1-\frac{v^2}{c^2}}{\left(1+\frac{u'v}{c^2}\right)^2}\frac{du'}{dt}.$$

But $u'=0$ instantaneously, and

$$\frac{du'}{dt} = \frac{du'}{dt'}\underbrace{\frac{dt'}{dt}}_{1/\gamma}$$, since t' is proper time

Finally, $$\frac{du}{dt} = \left(1-\frac{v^2}{c^2}\right)^{3/2}\underbrace{\frac{du'}{dt'}}_{= g}$$

Note that $u=v$ if $u'=0$ $\Leftarrow$ follows from (*)

Then $\displaystyle\int^{u} \frac{du}{\left(1-\frac{u^2}{c^2}\right)^{3/2}} = \int^{t} g\,dt$, or

$$\frac{u}{\sqrt{1-\frac{u^2}{c^2}}} = gt \Rightarrow u = \frac{gt}{\sqrt{1+\left(\frac{gt}{c}\right)^2}}$$

Now,

$$\int^{t'} dt' = \int^{t} \frac{dt}{\gamma} = \int^{t} \sqrt{1-\frac{u^2}{c^2}}\,dt \quad , \text{ or}$$

$$t' = \int^{t} dt \sqrt{1 - \frac{g^2t^2}{c^2\left(1+\frac{g^2t^2}{c^2}\right)^2}} =$$

$$= \int^{t} dt \frac{1}{\sqrt{1+\left(\frac{gt}{c}\right)^2}} = \frac{c}{g}\sinh^{-1}\left(\frac{gt}{c}\right).$$

Finally,

$$t = \frac{c}{g}\sinh\left(\frac{gt'}{c}\right)$$

$t' = 5\text{ yr} \quad \Rightarrow \quad t \simeq 86\text{ yr}$

$g = 9.86\ \frac{m}{s^2}$

$c = 3\times10^8\ \frac{m}{s}$

The total journey is $4t = 344$ yr.

Thus the year on Earth is $\underline{\underline{2444}}$.

b) In the first leg, the rocket ship traveled:

$$L = \int_0^{t=86\,yr \equiv T} u\,dt = \int_0^T \frac{gt}{\sqrt{1+\left(\frac{gt}{c}\right)^2}}\,dt =$$

$$= \frac{c^2}{g}\left[\sqrt{1+\underbrace{\frac{g^2T^2}{c^2}}_{\gg 1}} - 1\right] \simeq cT - \frac{c^2}{g} \simeq$$

$$\simeq 85 \text{ light-years}$$

The total distance in one dir'n is $2L = 170$ lyr.

Note that a beam of light would travel 172 lyr in 172 yr, not much further!

(11.1) a) $y' = y$ (motion in x direction)
$z' = z$

Homogeneity of space-time $\Rightarrow$ linear transform:

$$\begin{cases} x' = f_1 x + f_2 t \\ t' = g_1 x + g_2 t \end{cases}$$

f_1, f_2, g_1, g_2 do not depend on x & t but are otherwise arbitrary functions of v. Inspired by Galilean relativity, we take v out as follows:

$$\begin{cases} x' = f_1 x \overbrace{- v \tilde{f}_2}^{f_2} t \\ t' = g_2 t \underbrace{- v \tilde{g}_1}_{g_1} x \end{cases} \Leftarrow \text{expect } x' = x - vt \text{ as } \frac{v}{c} \to 0$$

But due to homogeneity the points along x cannot spread out as time increases $\Rightarrow f_1 = \tilde{f}_2$.

Moreover, due to isotropy expect $f_1, g_2, \tilde{g}_1$ to be functions of v^2 (no dependence on sign of v)

Finally,

$$\begin{cases} x' = f_1(v^2) x - f_1(v^2) v t, \\ t' = g_2(v^2) t - \tilde{g}_1(v^2) v x, \end{cases} \quad y' = y, \quad z' = z$$

The inverse can be obtained by $v \to -v$:

$$\begin{cases} x = f_1(v^2)\,x' + f_1(v^2)\,v\,t', & y = y', \\ t = g_2(v^2)\,t' + \tilde{g}_1(v^2)\,v\,x', & z = z'. \end{cases}$$

b) Transform $K \to K'$ & then $K' \to K$, should recover original coordinates:

$$x = f_1 x' + f_1 v t' = f_1(f_1 x - f_1 v t) +$$

$$+ f_1 v (g_2 t - \tilde{g}_1 v x) =$$

$$= \underbrace{(f_1^2 - f_1 \tilde{g}_1 v^2)}_{=1} x + \underbrace{(f_1 v g_2 - f_1^2 v)}_{=0} t$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$f_1(f_1 - v^2 \tilde{g}_1) = 1 \qquad\qquad f_1 = g_2$$

Likewise,

$$t = g_2 t' + \tilde{g}_1 v x' = g_2(g_2 t - \tilde{g}_1 v x) +$$

$$+ \tilde{g}_1 v (f_1 x - f_1 v t) =$$

$$= \underbrace{(\tilde{g}_1 v f_1 - \tilde{g}_1 v g_2)}_{=0} x + \underbrace{(g_2^2 - \tilde{g}_1 f_1 v^2)}_{=1} t$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$f_1 = g_2 \qquad\qquad g_2^2 = 1 + \tilde{g}_1 f_1 v^2$$

$f_1^2 - \tilde{g}_1 f_1 v^2 = 1$, same as above

So now $\begin{cases} \tilde{g}_1 = \dfrac{f_1^2-1}{f_1 v^2}, \\ g_2 = f_1 . \end{cases}$

The transform only depends on $f_1(v^2)$.

c) $u = \underset{\substack{\nearrow \\ \text{rename } f_1 \to f \text{ for simplicity}}}{\dfrac{dx}{dt}} = \dfrac{f(dx'+v\,dt')}{f\,dt' + v\,dx'\,\dfrac{f^2-1}{f v^2}} \,(=)$

$$u' = \frac{dx'}{dt'}$$

$$(=)\ \frac{u'+v}{1+v u' \underbrace{\frac{f-1/f}{f v^2}}_{\tilde{g}_1/f}} .$$

Universal limiting speed c:

$$c = \frac{c+v}{1+vc\,\dfrac{f-1/f}{f v^2}} \Rightarrow f = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma$$

Finally, $\begin{cases} x' = \gamma(x - vt), \\ t' = \gamma\underbrace{\left(t - \dfrac{vx}{c^2}\right)}_{t - \frac{\beta x}{c}}, \end{cases}$ $\quad y'=y, \quad z'=z.$

$$\frac{f^2-1}{fv} = \frac{\gamma^2-1}{\gamma v} = \frac{\gamma v}{c^2}\left(\frac{\gamma - 1/\gamma}{\gamma\left(\frac{v}{c}\right)^2}\right) = \frac{\gamma v}{c^2}\,\frac{\gamma^2-1}{\gamma^2\left(\frac{v}{c}\right)^2} = \frac{\gamma v}{c^2} .$$

Likewise,

$$\begin{cases} x = \gamma(x' + v t'), & y = y', \\ t = \gamma\left(t' + \dfrac{v x'}{C^2}\right), & z = z'. \end{cases}$$

$v \to -v$

$C = c$ (speed of light)
by experiments

11.3 Consider two Lorentz boosts along x-axis:

$$A_1 = \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \beta_1 = \frac{v_1}{c} \qquad \gamma_1 = \frac{1}{\sqrt{1-\beta_1^2}}$$

and same for A_2 (with β_2, γ_2).

Then

$$A_2 A_1 = \begin{pmatrix} \gamma_1\gamma_2(1+\beta_1\beta_2) & -\gamma_1\gamma_2(\beta_1+\beta_2) & 0 & 0 \\ -\gamma_1\gamma_2(\beta_1+\beta_2) & \gamma_1\gamma_2(1+\beta_1\beta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that

$$\gamma_1\gamma_2(1+\beta_1\beta_2) = \frac{1}{\sqrt{\frac{(1-\beta_1^2)(1-\beta_2^2)}{(1+\beta_1\beta_2)^2}}} =$$

$$= \frac{1}{\sqrt{1-\left(\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right)^2}} = \frac{1}{\sqrt{1-\frac{1}{c^2}\underbrace{\left(\frac{v_1+v_2}{1+\frac{v_1v_2}{c^2}}\right)}_{\equiv v}{}^2}} =$$

$$= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \equiv \gamma . \qquad \beta = \frac{v}{c}$$

Likewise,

$$-\gamma_1\gamma_2(\beta_1+\beta_2) = -\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\overbrace{\gamma_1\gamma_2(1+\beta_1\beta_2)}^{\gamma} =$$

$$= -\frac{v}{c}\gamma = -\beta\gamma$$

Finally,

$$A_2 A_1 = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This is a Lorentz boost with

$$v = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}.$$

Note that $v \neq v_1 + v_2$ at relativistic speeds