$\boxed{\text{HW \#6}}$

(10.1) a) Recall that for a perfectly conducting sphere,

$$\frac{d\sigma}{d\Omega}(\vec{n},\vec{\epsilon};\vec{n}_0,\vec{\epsilon}_0) = k^4 a^6 \left| \vec{\epsilon}^{\,*}\cdot\vec{\epsilon}_0 - \right.$$
$$\left. -\frac{1}{2}(\vec{n}\times\vec{\epsilon}^{\,*})\cdot(\vec{n}_0\times\vec{\epsilon}_0)\right|^2$$

(10.14)

Here, $\vec{\epsilon}$ is the outgoing polarization. To sum over outgoing polarizations, I'll introduce a basis $\perp \vec{n}$:

$$\begin{cases} \vec{\epsilon}_1 = \dfrac{\vec{n}\times\vec{n}_0}{\sin\theta}, \\ \vec{\epsilon}_2 = \underbrace{\vec{n}\times\vec{\epsilon}_1}_{\perp\vec{\epsilon}_1} = \dfrac{\vec{n}(\vec{n}\cdot\vec{n}_0)-\vec{n}_0}{\sin\theta} \end{cases}$$

$\sin\theta$ is the normalization factor; $\theta = \widehat{\vec{n}\vec{n}_0}$.

Note that $\sin^2\theta = \underbrace{1-(\vec{n}\cdot\vec{n}_0)^2}_{(\vec{n}\times\vec{n}_0)\cdot(\vec{n}\times\vec{n}_0)}$

Now, $$\frac{d\sigma}{d\Omega}(\vec{n};\vec{n}_0,\vec{\epsilon}_0) \overset{\text{sum over polar'ns}}{=} \frac{k^4a^6}{1-(\vec{n}\cdot\vec{n}_0)^2}\times$$
$$\times\left\{\left|\overset{\vec{\epsilon}_1}{(\vec{n}\times\vec{n}_0)}\cdot\vec{\epsilon}_0 - \frac{1}{2}(\vec{n}\times(\vec{n}\times\vec{n}_0))\cdot(\vec{n}_0\times\vec{\epsilon}_0)\right|^2 + \right.$$
$$\left. + \left|\underset{\uparrow\vec{\epsilon}_2}{(\vec{n}(\vec{n}\cdot\vec{n}_0)-\vec{n}_0)}\cdot\vec{\epsilon}_0 - \frac{1}{2}(\vec{n}\times(\vec{n}(\vec{n}\cdot\vec{n}_0)-\vec{n}_0))\cdot(\vec{n}_0\times\vec{\epsilon}_0)\right|^2\right\} \ominus$$

$$\stackrel{\text{vector algebra using } \vec{n}_0\cdot\vec{\epsilon}_0=0}{=} \frac{k^4 a^6}{1-(\vec{n}\cdot\vec{n}_0)^2}\left[|\vec{n}\cdot(\vec{n}_0\times\vec{\epsilon}_0)|^2 \left[1-\frac{\vec{n}\cdot\vec{n}_0}{2}\right]^2 + \right.$$

$$\left. + |\vec{n}\cdot\vec{\epsilon}_0|^2\left[\frac{1}{2}-(\vec{n}\cdot\vec{n}_0)\right]^2\right].$$

Expand the squares & simplify:

$$\frac{d\sigma}{d\Omega}(\vec{n};\vec{n}_0,\vec{\epsilon}_0) = \frac{k^4 a^6}{1-(\vec{n}\cdot\vec{n}_0)^2} \times$$

$$\times\Big[\left(\frac{5}{4}-\vec{n}\cdot\vec{n}_0\right)\left(|\vec{n}\cdot(\vec{n}_0\times\vec{\epsilon}_0)|^2 + |\vec{n}\cdot\vec{\epsilon}_0|^2\right) -$$

$$- [1-(\vec{n}\cdot\vec{n}_0)^2]\left[\frac{1}{4}|\vec{n}\cdot(\vec{n}_0\times\vec{\epsilon}_0)|^2 + |\vec{n}\cdot\vec{\epsilon}_0|^2\right]\Big].$$

To proceed, note that

$\vec{n}_0, \vec{\epsilon}_0, \vec{n}_0\times\vec{\epsilon}_0$ form a basis.

$\vec{n}$ can be expanded in this basis:

$\vec{n}\cdot\vec{n}_0$, $\vec{n}\cdot\vec{\epsilon}_0$, $\vec{n}\cdot(\vec{n}_0\times\vec{\epsilon}_0)$.

$$\vec{n}^2 = 1 \implies |\vec{n}\cdot(\vec{n}_0\times\vec{\epsilon}_0)|^2 + |\vec{n}\cdot\vec{\epsilon}_0|^2 =$$

$$= 1-(\vec{n}\cdot\vec{n}_0)^2.$$

Finally,

$$\frac{d\sigma}{d\Omega}(\vec{n};\vec{n}_0,\vec{\epsilon}_0) = k^4 a^6\left[\frac{5}{4}-\vec{n}\cdot\vec{n}_0 - |\vec{n}\cdot\vec{\epsilon}_0|^2 - \right.$$

$$\left. - \frac{1}{4}|\vec{n}\cdot(\vec{n}_0\times\vec{\epsilon}_0)|^2\right],$$ as desired

b) As in (a), expand $\vec{n}$ in the $\vec{n}_0, \vec{\epsilon}_0, \vec{n}_0 \times \vec{\epsilon}_0$ basis:

$$\begin{cases} \vec{n}\cdot\vec{n}_0 = \cos\theta, \\ \vec{n}\cdot\vec{\epsilon}_0 = \sin\theta\cos\varphi, \\ \vec{n}\cdot(\vec{n}_0\times\vec{\epsilon}_0) = \sin\theta\sin\varphi. \end{cases}$$

Then

$$\frac{d\sigma}{d\Omega}(\theta,\varphi) \underset{\text{from (a)}}{=} k^4 a^6 \left[\frac{5}{4} - \cos\theta - \sin^2\theta\cos^2\varphi - \frac{1}{4}\sin^2\theta\sin^2\varphi \right] =$$

$$= k^4 a^6 \left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos 2\varphi \right].$$

c) First, $\frac{d\sigma}{d\Omega}(\frac{\pi}{2},\varphi) = k^4 a^6 \left[\frac{5}{8} - \frac{3}{8}\cos 2\varphi \right]$.

Then
$$\begin{cases} \frac{d\sigma}{d\Omega}(\frac{\pi}{2}, 0) = \frac{k^4 a^6}{4}, \\ \frac{d\sigma}{d\Omega}(\frac{\pi}{2},\frac{\pi}{2}) = k^4 a^6. \end{cases}$$

The ratio is $1/4$.

$$\theta = \frac{\pi}{2}, \varphi = 0 \Rightarrow \begin{cases} \vec{n}\cdot\vec{n}_0 = 0, \\ \vec{n}\cdot\vec{\epsilon}_0 = 1, \\ \vec{n}\cdot(\vec{n}_0\times\vec{\epsilon}_0) = 0 \end{cases} \Rightarrow \vec{n} \uparrow\uparrow \vec{\epsilon}_0$$

This means that the induced electric dipole (which is $\uparrow\uparrow \vec{E}_0$), is $\uparrow\uparrow \vec{n}$ as well. But the dipole does not radiate along its axis, so radiation is purely magnetic dipole in this case.

Similarly, at $\theta = \frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$ radiation is purely electric dipole in nature. The factor of $1/4$ indicates the relative strength of magnetic dipole / electric dipole radiation.

(10.4) a) $R \ll \lambda \Rightarrow$ we are again in the quasistatic limit of a sphere in an external uniform field.

The induced electric dipole moment is:

$$\vec{p} = 4\pi\epsilon_0 \left(\frac{\epsilon_r^c - 1}{\epsilon_r^c + 2} \right) R^3 \vec{\epsilon}_0 E_0$$

(10.5)

Since the sphere is only slightly lossy, we can neglect the induced magnetic dipole moment.

Then $\underset{\text{complex}}{\epsilon_r^c} = \underset{\text{real}}{\epsilon_r} + i \frac{\overset{\text{real}}{\sigma}}{\omega \epsilon_0}$

$$\omega = ck = \frac{k}{\epsilon_0 Z_0} \Rightarrow \epsilon_r^c = \epsilon_r + i \frac{\sigma}{k} Z_0$$

Now recall that

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0)^2 E_0^2} |\vec{\epsilon}^* \cdot \vec{p}|^2 \text{, where}$$

(10.4)

$$\vec{p} = 4\pi\epsilon_0 \frac{(\epsilon_r - 1) + i\frac{\sigma Z_0}{k}}{(\epsilon_r + 2) + i\frac{\sigma Z_0}{k}} R^3 \vec{\epsilon}_0 E_0 .$$

Then $$\frac{d\sigma}{d\Omega} = \frac{(\epsilon_r - 1)^2 + \left(\frac{Z_0\sigma}{k}\right)^2}{(\epsilon_r + 2)^2 + \left(\frac{Z_0\sigma}{k}\right)^2} k^4 R^6 |\vec{\epsilon}^* \cdot \vec{\epsilon}_0|^2 .$$

For unpolarized light, we have

$$\frac{d\sigma_{unpol}}{d\Omega} = \frac{(\epsilon_r - 1)^2 + \left(\frac{Z_0 \sigma}{k}\right)^2}{(\epsilon_r + 2)^2 + \left(\frac{Z_0 \sigma}{k}\right)^2} k^4 R^6 \, \frac{1 + \cos^2\theta}{2}$$

(10.10)

Finally,

$$\sigma_{unpol} = \int d\Omega \, \frac{d\sigma}{d\Omega} = \frac{8\pi}{3} \frac{(\epsilon_r - 1)^2 + \left(\frac{Z_0 \sigma}{k}\right)^2}{(\epsilon_r + 2)^2 + \left(\frac{Z_0 \sigma}{k}\right)^2} k^4 R^6 .$$

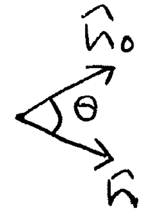
(10.11)

(10.9) a) Use the 1st Born appr'n (ϵ_r close to 1):

$$\frac{\vec{\epsilon}^* \cdot \vec{A}^{(1)}_{sc}}{D_0} = \frac{k^2}{4\pi} \int d^3x \, e^{i\vec{q}\vec{x}} \left[\vec{\epsilon}^* \cdot \vec{\epsilon}_0 \frac{\delta\epsilon}{\epsilon_0} + \right.$$

$$\left. + (\hat{n}_0 \times \vec{\epsilon}_0)(\hat{n} \times \vec{\epsilon}^*) \frac{\delta\mu}{\mu_0} \right] \qquad (10.31)$$

Here, $\vec{q} = k(\hat{n}_0 - \hat{n})$, so that



$q = 2k \sin\frac{\theta}{2}$

$$q^2 = 2k^2(1 - \cos\theta) = 4k^2 \sin^2\frac{\theta}{2}.$$

Further, $\delta\mu = 0$ and

$$\frac{\delta\epsilon}{\epsilon_0} = \begin{cases} \epsilon_r - 1 & r \le a \\ 0 & r > a \end{cases}$$

Then

$$\frac{\vec{\epsilon}^* \cdot \vec{A}^{(1)}_{sc}}{D_0} = \frac{k^2}{4\pi}(\epsilon_r - 1)(\vec{\epsilon}^* \cdot \vec{\epsilon}_0) \times$$

$$\times \underbrace{\int_{r<a} d^3x \, e^{i\vec{q}\vec{x}}}_{I}.$$

$$I = 2\pi \int_0^a dr \, r^2 \int_{-1}^1 dx \, e^{iqrx} =$$

$$= \frac{4\pi}{q} \int_0^a dr \, r \sin(qr) = \frac{4\pi}{q^3} [\sin(qa) - qa\cos(qa)].$$

Consequently,

$$\frac{\vec{\epsilon}^{*}\cdot\vec{A}_{sc}^{(1)}}{D_0} = \frac{(ka)^2}{q}(\epsilon_r-1)(\vec{\epsilon}^{*}\cdot\vec{\epsilon}_0)\times$$

$$\times\underbrace{\frac{\sin(qa)-qa\cos(qa)}{(qa)^2}}_{j_1(qa)\ -\ \text{Bessel f'n with } \ell=1}$$

Then $\frac{d\sigma}{d\Omega} = \left|\frac{\vec{\epsilon}^{*}\cdot\vec{A}_{sc}^{(1)}}{D_0}\right|^2 = k^4a^6|\epsilon_r-1|^2\times$

$$\times|\vec{\epsilon}^{*}\cdot\vec{\epsilon}_0|^2\left(\frac{j_1(qa)}{qa}\right)^2.$$

The unpolarized cross-section is

$$\frac{d\sigma^{unp}}{d\Omega} = k^4a^6|\epsilon_r-1|^2\left(\frac{j_1(qa)}{qa}\right)^2\frac{1+\cos^2\theta}{2}$$

Note that if $ka \ll 1 \Rightarrow qa \ll 1$,

and $j_1(qa) \to \frac{qa}{3}$ as $qa \to 0$.

So $$\frac{d\sigma^{unp}}{d\Omega}\xrightarrow[qa\ll 1]{}\frac{k^4a^6}{9}|\epsilon_r-1|^2\,\frac{1+\cos^2\theta}{2}.$$

If $ka \gg 1 \Rightarrow qa = 2ka\sin\frac{\theta}{2} \gg 1$, true for all angles away from $\theta=0$

and $j_1(qa)\xrightarrow[qa\gg 1]{} -\frac{\cos(qa)}{qa}$

So, for θ away from 0

$$\frac{d\sigma^{unp}}{d\Omega} \underset{qa\gg 1}{\sim} \frac{1}{(qa)^4} = \underbrace{\frac{1}{(2ka\sin\frac{\theta}{2})^4}}_{\text{falls off rapidly}}$$

In σ^{unp}, $\int d\Omega$ will be highly peaked around $\theta=0$:

$$\sigma^{unp} \sim \int d\Omega \underbrace{\frac{1+\cos^2\theta}{2}}_{\simeq 1} \left(\frac{j_1(qa)}{qa}\right)^2 \simeq$$

$$\simeq 2\pi \int_0^{\pi} \sin\theta\, d\theta \left(\frac{j_1(qa)}{qa}\right)^2 \simeq 2\pi \int_0^{\pi} d\theta\, \theta \left(\frac{j_1(ka\theta)}{ka\theta}\right)^2 \simeq$$

$qa \simeq ka\theta$ if θ is small

extend to $+\infty$

$$\simeq \frac{2\pi}{(ka)^2} \int_0^{\infty} j_1^2(x)\,\frac{dx}{x} = \frac{\pi}{2(ka)^2}.$$

Finally,

$$\sigma^{unp} \simeq \frac{\pi a^2}{2}(ka)^2\,|\epsilon_r - 1|^2, \text{ as desired}$$

Note that $ka \ll 1 \Rightarrow \sigma^{unp} \sim k^4$

$ka \gg 1 \Rightarrow \sigma^{unp} \sim k^2.$