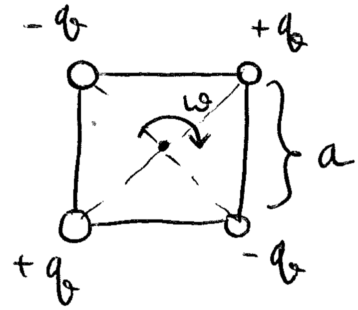
HW#5 Solutions 30/30

9.2



Choose the z-axis as the axis of rotation. Choose $t=0$ s.t. the phase is ωt for the $+q$ charge in the upper right corner.

Then
$$\rho(\vec{x},t) = q\,\delta(z)\Big\{\delta(x-\underbrace{\tfrac{a}{\sqrt{2}}}_{\frac{1}{2}\text{ the diagonal}}\cos\omega t)\,\delta(y-\tfrac{a}{\sqrt{2}}\sin\omega t)-$$
$$-\delta(x-\tfrac{a}{\sqrt{2}}\cos(\omega t+\tfrac{\pi}{2}))\,\delta(y-\tfrac{a}{\sqrt{2}}\sin(\omega t+\tfrac{\pi}{2}))+$$
$$+\delta(x-\tfrac{a}{\sqrt{2}}\cos(\omega t+\pi))\,\delta(y-\tfrac{a}{\sqrt{2}}\sin(\omega t+\pi))-$$
$$-\delta(x-\tfrac{a}{\sqrt{2}}\cos(\omega t+\tfrac{3\pi}{2}))\,\delta(y-\tfrac{a}{\sqrt{2}}\sin(\omega t+\tfrac{3\pi}{2}))\Big\}=$$

$$= q\,\delta(z)\Big\{\delta(x-\tfrac{a}{\sqrt{2}}\cos\omega t)\,\delta(y-\tfrac{a}{\sqrt{2}}\sin\omega t)-$$
$$-\delta(x+\tfrac{a}{\sqrt{2}}\sin\omega t)\,\delta(y-\tfrac{a}{\sqrt{2}}\cos\omega t)+$$
$$+\delta(x+\tfrac{a}{\sqrt{2}}\cos\omega t)\,\delta(y+\tfrac{a}{\sqrt{2}}\sin\omega t)-$$
$$-\delta(x-\tfrac{a}{\sqrt{2}}\sin\omega t)\,\delta(y+\tfrac{a}{\sqrt{2}}\cos\omega t)\Big\}$$

Recall that $Q_{\alpha\beta} = \int dV\,\rho(\vec{x},t)\,[3x_\alpha x_\beta - r^2\delta_{\alpha\beta}]$

Now, $\int dV\,\rho(\vec{x},t)\,z^2 = 0$,

$$\int dV\,\rho(\vec{x},t)\,x^2 = q\Big\{\frac{a^2}{2}\cos^2\omega t - \frac{a^2}{2}\sin^2\omega t + \frac{a^2}{2}\cos^2\omega t - \frac{a^2}{2}\sin^2\omega t\Big\} =$$

$$= q a^2\cos(2\omega t).$$

Likewise, $\int dV\, \rho(\vec{x},t)\, y^2 = -q a^2 \cos(2\omega t)$.

$\int dV\, \rho(\vec{x},t)\, xz = \int dV\, \rho(\vec{x},t)\, yz = 0$.

Finally, $\int dV\, \rho(\vec{x},t)\, xy = q \Big\{ \frac{a^2}{2} \sin\omega t \cos\omega t +$

$+ \frac{a^2}{2} \sin\omega t \cos\omega t + \frac{a^2}{2} \sin\omega t \cos\omega t +$

$+ \frac{a^2}{2} \sin\omega t \cos\omega t \Big\} = q a^2 \sin(2\omega t)$.

Thus $Q_{11} = \int dV\, \rho\, (2x^2 - y^2 - z^2) = 3 q a^2 \cos(2\omega t) =$

$= 3 q a^2 \, \mathrm{Re}\{ e^{-2i\omega t} \}$

$Q_{22} = \int dV\, \rho\, (2y^2 - x^2 - z^2) = -3 q a^2 \cos(2\omega t) =$

$= -3 q a^2 \, \mathrm{Re}\{ e^{-2i\omega t} \}$

$Q_{12} = Q_{21} = \int dV\, \rho\, xy = 3 q a^2 \sin(2\omega t) =$

$= 3 q a^2 \, \mathrm{Re}\{ i e^{-2i\omega t} \}$.

All other $Q_{\alpha\beta} = 0$.

Therefore, $Q_{\alpha\beta} = 3 q a^2 \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{-2i\omega t}$

The freq. of radiation is 2ω $\left(k = \frac{2\omega}{c}\right)$, consistent with the symmetry of the charge distribution.

This system has no electric dipole moment (2 ↑↓ dipoles) , no magnetic dipole moment ($\langle \vec{J} \rangle = 0$). Thus radiation is

average over $T = \frac{2\pi}{\omega}$

dominated by the electric quadrupole:

$$\vec{H} = -\frac{ick^3}{24\pi} \frac{e^{ikr}}{r} \vec{n} \times \vec{Q}(\vec{n})$$

(9.44)

$$\vec{n} = (\sin\theta \cos\varphi, \sin\theta \cos\varphi, \cos\theta)$$

$$\vec{Q}(\vec{n}) = (Q_{1\alpha} n_\alpha , Q_{2\alpha} n_\alpha , Q_{3\alpha} n_\alpha) =$$

$$= \big(3qa^2 \sin\theta(\cos\varphi + i\sin\varphi), 3qa^2 \sin\theta(i\cos\varphi - \sin\varphi),$$
$$0\big) = \underline{\underline{3qa^2 \sin\theta\, e^{i\varphi} (\hat{x} + i\hat{y})}}.$$

Thus
$$\begin{cases} \vec{H} = -\frac{ick^3}{8\pi}(qa^2)\frac{e^{ikr}}{r} e^{i\varphi} \sin\theta\, * \\ \quad * (-i\cos\theta, \cos\theta, i\sin\theta\, e^{i\varphi}), \\ \vec{E} = Z_0 \vec{H} \times \vec{n}. \end{cases}$$

[$e^{-2i\omega t}$ dependence is implied]

Finally, $$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{1152\pi^2} k^6 \underbrace{|(\vec{n}\times\vec{Q})\times\vec{n}|^2}_{|\vec{Q}|^2 - |\vec{Q}\cdot\vec{n}|^2} =$$

(9.45)

$$= \frac{c^2 Z_0}{1152\pi^2} k^6 (3qa^2)^2 \underbrace{\sin^2\theta\,(2-\sin^2\theta)}_{} =$$

$$(1-\cos^2\theta)(1+\cos^2\theta) =$$

$$= (1-\cos^4\theta)$$

$$= \frac{c^2 Z_0 k^6 (qa^2)^2}{128\pi^2} (1-\cos^4\theta).$$

$$k = \frac{2\omega}{c}$$

The total radiated power is

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{Z_0 \omega^6}{2\pi^2 c^4} (qa^2)^2 \int d\Omega\,(1-\cos^4\theta) =$$

$$= \frac{8 Z_0 \omega^6}{5\pi c^4} q^2 a^4 .$$

(9.3) This setup (two half-spheres separated by a gap) will produce a non-zero electric dipole, which will dominate in the $kd \gg 1$ limit.

Eq'n just above (3.38):

$$\varphi(r,\theta,t) = \frac{V(t)}{\sqrt{\pi}} \sum_{j=1}^{\infty} (-1)^{j+1} \frac{(2j-\frac{1}{2})\Gamma(j-\frac{1}{2})}{j!} \left(\frac{a}{r}\right)^{2j} P_{2j-1}(\cos\theta),$$

↑ scalar potential

where a - sphere radius, and $P_\ell(x)$ is the Legendre polynomial of order ℓ.

$j=1$ is the electric dipole term:

$$\varphi_1(r,\theta,t) = \frac{V(t)}{\sqrt{\pi}} \frac{3}{2} \underbrace{\Gamma\left(\tfrac{1}{2}\right)}_{=\sqrt{\pi}} \left(\frac{R}{r}\right)^2 \underbrace{P_1(\cos\theta)}_{=\cos\theta} =$$

sphere radius

$$= \frac{3V}{2}\left(\frac{R}{r}\right)^2 \cos\theta \cos\omega t \equiv$$

$$\equiv \frac{p}{4\pi\epsilon_0 r^2} \cos\theta \cos\omega t,$$

← scalar potential due to dipole pointing in $\hat{z}$-direction

s.t. the dipole moment

$$p = 6\pi\epsilon_0 V R^2, \quad \vec{p} = p\hat{z}.$$

$\hat{z}$ is $\perp$ to the plane of the insulating gap.

For electric dipole radiation, everything is known (see Ch. 9):

radiation zone

$k = \frac{\omega}{c}$

$$\left\{\begin{aligned} \vec{H} &= -\frac{3V}{2Z_0}(kR)^2 \frac{e^{ikr}}{r} \sin\theta\, \hat{\varphi}, \\ \vec{E} &= -\frac{3V}{2}(kR)^2 \frac{e^{ikr}}{r} \sin\theta\, \hat{\theta}, \\ \frac{dP}{d\Omega} &= \frac{9V^2}{8Z_0}(kR)^4 \sin^2\theta, \\ P &= 3\pi (kR)^4 \frac{V^2}{Z_0}. \end{aligned}\right.$$

9.5 (a) Recall that with time variation $e^{-i\omega t}$ and in the Lorenz gauge $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0$, the wave equations

$$\begin{cases} \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}, \\ \nabla^2 \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \end{cases}$$

are solved by $\begin{cases} \vec{A}(\vec{x},t) = \vec{A}(\vec{x})\, e^{-i\omega t}, \\ \varphi(\vec{x},t) = \varphi(\vec{x})\, e^{-i\omega t} \end{cases}$, where

$\vec{J}(\vec{x},t) = \vec{J}(\vec{x})\, e^{-i\omega t}$

$$\begin{cases} \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')\, e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}, \\ \varphi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec{x}')\, e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \end{cases}$$

In the $r \gg \underbrace{d}_{\text{spatial extent of the system}}$ limit,

$|\vec{x}-\vec{x}'| \simeq r - r'\,\vec{n}\cdot\vec{n}'$, and $\begin{cases} \vec{x} = r\vec{n}, \\ \vec{x}' = r'\vec{n}'. \end{cases}$

$$\frac{1}{|\vec{x}-\vec{x}'|} \simeq \frac{1}{r} + \frac{r'}{r^2}\,\vec{n}\cdot\vec{n}'.$$

In the $\lambda \gg d$ (long-wavelength) limit, we can also expand

$$e^{-ikr'(\vec{n}\cdot\vec{n}')} \simeq 1 - ikr'(\vec{n}\cdot\vec{n}') + \ldots$$

Then $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\Big\{\int d^3x'\,\vec{J}(\vec{x}')\left[1+\frac{r'}{r}\vec{n}\cdot\vec{n}'\right] -$

$- ik\int d^3x'\,\vec{J}(\vec{x}')\left[1+\frac{r'}{r}\vec{n}\cdot\vec{n}'\right](r'(\vec{n}\cdot\vec{n}')) + \ldots\Big\} \cong$

$\cong \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,\vec{J}(\vec{x}') = -\frac{i\mu_0\omega}{4\pi}\frac{e^{ikr}}{r}\vec{p}$,

electric dipole appr'n

shown in class, see (9.16) also

where $\vec{p} = \int d^3x'\,\rho(\vec{x}')\,\vec{x}'$ is the electric dipole moment.

Likewise,

$\varphi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\Big\{\int d^3x'\,\rho(\vec{x}')\left[1+\frac{r'}{r}\vec{n}\cdot\vec{n}'\right] -$

$- ik\int d^3x'\,\rho(\vec{x}')\left[1+\frac{r'}{r}\vec{n}\cdot\vec{n}'\right](r'(\vec{n}\cdot\vec{n}')) + \ldots\Big\} \cong$

the monopole term only contributes at $\omega = 0$, so we discard it here ...

$\cong \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r^2}\vec{n}\cdot\vec{p}\,(1-ikr)$.

Note that eq's for $\vec{A}(\vec{x})$ & $\varphi(\vec{x})$ are valid in all zones.

(b) $\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A} = - \frac{i\omega}{4\pi} \vec{\nabla} \times \left(\frac{e^{ikr}}{r} \vec{p} \right) \ominus$

Note that $e_{ijk} \partial_j (f(r) p_k) = e_{ijk} \left(n_j \frac{\partial f(r)}{\partial r} \right) p_k$, s.t.

$$\vec{\nabla} \times (f(r) \vec{p}) = (\vec{n} \times \vec{p}) \frac{\partial f(r)}{\partial r}$$

$$\ominus - \frac{i\omega}{4\pi} (\vec{n} \times \vec{p}) \frac{\partial}{\partial r} \left(\frac{e^{ikr}}{r} \right) =$$

$$= - \frac{i\omega}{4\pi} (\vec{n} \times \vec{p}) \left(ik - \frac{1}{r} \right) \frac{e^{ikr}}{r} =$$

$$= \frac{ck^2}{4\pi} (\vec{n} \times \vec{p}) \left(1 - \frac{1}{ikr} \right) \frac{e^{ikr}}{r}.$$

Likewise,

$$\vec{E} = -\vec{\nabla} \varphi - \frac{\partial \vec{A}}{\partial t} = - \frac{1}{4\pi \epsilon_0} \left[\vec{\nabla} \left(\frac{e^{ikr}}{r^2} \vec{n} \cdot \vec{p} \right) - \right.$$

$$\left. - ik \vec{\nabla} \left(\frac{e^{ikr}}{r} \vec{n} \cdot \vec{p} \right) \right] + \frac{\omega^2 \mu_0}{4\pi} \frac{e^{ikr}}{r} \vec{p} \boxminus$$

Use $\vec{\nabla} [f(r) \vec{n} \cdot \vec{p}] = \frac{f(r)}{r} \vec{p} + (\vec{n} \cdot \vec{p}) \vec{n} \left[\frac{\partial f}{\partial r} - \frac{f}{r} \right]$

$$\boxminus \frac{1}{4\pi \epsilon_0} e^{ikr} \left[\frac{k^2}{r} (\vec{n} \times \vec{p}) \times \vec{n} + [3\vec{n}(\vec{n} \cdot \vec{p}) - \vec{p}] \times \right.$$
$$\left. \times \left(\frac{1}{r^3} - \frac{ik}{r^2} \right) \right].$$