HW#4 solutions

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(7.12) (a) Work in ω-space.

Charge continuity eq'n:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{J}(\omega) = i\omega\rho(\omega)$$

$$\rho(t) = \frac{1}{\sqrt{2\pi}}\int d\omega\, e^{-i\omega t}\rho(\omega)$$

Ohm's law: $\vec{J}(\omega) = \sigma(\omega)\vec{E}(\omega)$

linear in ω-space

Then $\sigma(\omega)\,\vec{\nabla}\cdot\vec{E}(\omega) = i\omega\rho(\omega)$

spatially uniform conductor

Now, $\vec{\nabla}\cdot\vec{E}(\omega) = \dfrac{\rho(\omega)}{\epsilon_0}$ Coulomb's law in ω-space

assumed $\epsilon = \epsilon_0$

Then $(\sigma(\omega) - i\omega\epsilon_0)\rho(\omega) = 0$, as desired.

(b) Now use

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau},$$

where $\sigma_0 = \epsilon_0\omega_p^2\tau$ ← damping time

plasma freq.

In other words,

$$\left[\frac{\omega_p^2\tau}{1-i\omega\tau} - i\omega\right] p(\omega) = 0 .$$

If $p(\omega) \neq 0$ (non-trivial solution) $\Rightarrow \dfrac{\omega_p^2\tau}{1-i\omega\tau} = i\omega$, or

$$\omega^2 + \frac{i}{\tau}\omega - \omega_p^2 = 0 . \quad (*)$$

Solve $(*)$: $\omega_{1,2} = \dfrac{-i \pm \sqrt{4\omega_p^2\tau^2 - 1}}{2\tau}$.

If $\omega_p\tau \gg 1$, $\omega_{1,2} \simeq \pm\omega_p - \dfrac{i}{2\tau}$.

To summarize,

$$p(\omega) = \begin{cases} p^+, & \omega = \omega_p - \frac{i}{2\tau} \\ p^-, & \omega = -\omega_p - \frac{i}{2\tau} \\ 0, & \text{all other freqs.} \end{cases}$$

But then $p(t) = \left(p^+ e^{-i\omega_p t} + p^- e^{+i\omega_p t}\right) e^{-\frac{t}{2\tau}}$.

from FT

So, $p(t)$ oscillates with ω_p & decays $\sim e^{-t/2\tau}$.

Note that if $\sigma(\omega) \simeq \sigma(0) = \sigma_0$,

$\sigma_0 - i\omega\epsilon_0 = 0$ if $p(\omega) \neq 0$, and

$\omega = -i\dfrac{\sigma_0}{\epsilon_0}$. So this would have <u>no</u> oscillations & damping $\sim e^{-\frac{t\sigma_0}{\epsilon_0}}$.

(7.4) $\sigma; \epsilon$ free space / medium interface

(a) Normal incidence:

$$\frac{\overset{\text{reflected}}{E_0''}}{\underset{\text{incident}}{E_0}} = \frac{1-n}{1+n}$$

Recall that $n = c/v = c\sqrt{\mu\epsilon}$.

We have argued that for a conductor, we can neglect $\epsilon_b(\omega)$, yielding

$$\epsilon \simeq \frac{i\sigma}{\omega}$$ (true for low ω, or large σ)

Then $n = c\sqrt{i\frac{\mu\sigma}{\omega}} = (1+i)\underbrace{\frac{c}{\omega}}_{k^{-1}}\underbrace{\sqrt{\frac{\mu\sigma\omega}{2}}}_{\delta^{-1}} =$

$\sqrt{i} = \frac{1+i}{\sqrt{2}}$

$= (1+i)\frac{1}{k\delta}$, where $[\delta] = L$, δ - skin depth.

Finally, $$\frac{E_0''}{E_0} = \frac{1-(1+i)\frac{1}{k\delta}}{1+(1+i)\frac{1}{k\delta}} \equiv \underbrace{re^{i\varphi}}_{\text{complex}}$$

After some algebra,

$$\begin{cases} r = \dfrac{\sqrt{\omega^4\delta^4 + 4c^4}}{2c^2 + 2c\omega\delta + \omega^2\delta^2}, \\ \tan\varphi = -\dfrac{2c\omega\delta}{\omega^2\delta^2 - 2c^2}. \end{cases}$$

(b)

Note that $\delta = \sqrt{\frac{2}{\mu\sigma\omega}} \to 0$ as $\sigma \to +\infty$ (good conductor)

Then $r \to 1$, $\tan\varphi \to 0^+ \Rightarrow \varphi \to \pi$ ← phase change due to $n > 1$

The reflected wave must have a phase change since $n > 1$.

For a good conductor,

$$R = r^2 = \frac{\omega^4 \delta^4 + 4c^4}{(2c^2 + 2c\omega\delta + \omega^2\delta^2)^2} \approx$$

expand to $O(\delta)$

$$\approx 1 - 2\frac{\omega}{c}\delta .$$

For a poor conductor,

$\sigma \to 0$ & $\delta \to +\infty$.

In this limit, $\epsilon_b(\omega)$ cannot be neglected & $\epsilon = i\frac{\sigma}{\omega}$ breaks down. The medium behaves as a regular insulator.