HW #3 solutions

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13.2 Equal and Opposite Magnetization

(a) There is no free current. The magnetization in each region is uniform so the bulk magnetization current density $\mathbf{j}_M = \nabla \times \mathbf{M} = 0$. The magnetization is normal to the $z = 0$ interface so the surface magnetization current density $\mathbf{K} = \mathbf{M} \times \hat{\mathbf{n}} = 0$. There is no source current of any kind, so $\mathbf{B} = 0$ everywhere.

(b) There is no bulk magnetic charge $\rho^* = -\nabla \cdot \mathbf{M}$ but there is a surface charge density $\sigma^* = \mathbf{M} \cdot \hat{\mathbf{n}}$. There is a contribution $\sigma = M$ at $z = 0$ due to the $z > 0$ region. An identical contribution comes from the $z < 0$ region. Therefore, since an outward-pointing electric field $E = \sigma/2\epsilon_0$ is created by a planar surface density of electric charge σ, we get an outward-pointing field $H = M$ in this case. Since $\mathbf{M}$ points inward to the same interface, we conclude that $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = 0$ everywhere.

13.5 The Virtues of Magnetic Charge

(a) The text establishes that $\mathbf{m} = \int d^3r\, \mathbf{M}$. On the other hand, using the proposed formula, the k^{th} component of the magnetic dipole moment of the sample is

$$m_k = -\int d^3r\, r_k\, \nabla \cdot \mathbf{M} = -\int d^3r\, \nabla \cdot (\mathbf{M} r_k) + \int d^3r\, (\mathbf{M} \cdot \nabla) r_k = \int d^3r\, M_k.$$

(b) By definition, the interaction energy between two current distributions is

$$\hat{V}_B = -\frac{\mu_0}{4\pi}\int d^3r \int d^3r' \frac{\mathbf{j}_1(\mathbf{r})\cdot\mathbf{j}_2(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

Using the definition of the vector potential in the Coulomb gauge, this is

$$\hat{V}_B = -\int d^3r\, \mathbf{j}_1\cdot\mathbf{A}_2 = -\int d^3r\, \mathbf{A}_2\cdot\nabla\times\mathbf{M}_1 = \int d^3r\, \nabla\cdot(\mathbf{A}_2\times\mathbf{M}_1) - \int d^3r\, \mathbf{M}_1\cdot\nabla\times\mathbf{A}_2.$$

Finally, using the divergence theorem and the fact that $\mathbf{M}_1$ is zero on the integration surface at infinity, we conclude that

$$\hat{V}_B = -\int d^3r\, \mathbf{M}_1\cdot\mathbf{B}_2.$$

Precisely the same steps beginning with $\hat{V}_B = -\int d^3r\, \mathbf{j}_2\cdot\mathbf{A}_1$ establish the reciprocity relation.

(c) It is simplest to begin with the proposed formula and show that it is equivalent to the expression derived in part (b). Then, because $\mathbf{B}_2 = \mu_0\mathbf{H}_2$ in the part of space where $\mathbf{M}_1 \neq 0$,

$$\begin{aligned}
\hat{V}_B &= \frac{\mu_0}{4\pi}\int d^3r \int d^3r' \frac{\nabla\cdot\mathbf{M}_1(\mathbf{r})\nabla'\cdot\mathbf{M}_2(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\
&= \frac{\mu_0}{4\pi}\int d^3r'\, \nabla'\cdot\mathbf{M}_2(\mathbf{r}') \int d^3r \left\{\nabla\cdot\left[\frac{\mathbf{M}_1(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}\right] - \mathbf{M}_1(\mathbf{r})\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right\} \\
&= \int d^3r\, \mathbf{M}_1(\mathbf{r})\cdot\nabla\frac{\mu_0}{4\pi}\int d^3r' \frac{\rho_2^*(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\
&= -\int d^3r\, \mathbf{M}_1(\mathbf{r})\cdot\mu_0\mathbf{H}_2(\mathbf{r}) \\
&= -\int d^3r\, \mathbf{M}_1(\mathbf{r})\cdot\mathbf{B}_2(\mathbf{r}).
\end{aligned}$$

13.11 Lunar Magnetism

We have $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, where $\mathbf{H} = -\nabla\psi$ and ψ satisfies the Poisson-like equation

$$\nabla^2\psi = \nabla \cdot \mathbf{M}.$$

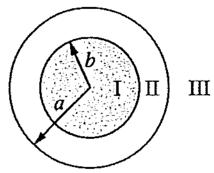
In addition, at the boundary between regions, it is necessary to satisfy the matching conditions

$$\psi_1(\mathbf{r}_S) = \psi_2(\mathbf{r}_S)$$

and

$$\left[\frac{\partial\psi_1}{\partial n_1} - \frac{\partial\psi_2}{\partial n_1}\right]_S = [\mathbf{M}_1 - \mathbf{M}_2]_S \cdot \hat{\mathbf{n}}_1.$$

We call the core, crust, and exterior of the Moon regions I, II, and III, respectively, as shown below.



The impressed magnetization $\mathbf{M}$ of the core is stated to be proportional to a dipole field $\mathbf{B}_\mathrm{d}$ centered at the origin. If we align the magnetic moment $\mathbf{m}$ with the z-axis,

$$\mathbf{B}_\mathrm{d}(r,\theta) = \frac{\mu_0 m}{4\pi}\frac{3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{z}}}{r^3} = \frac{\mu_0 m}{4\pi}\frac{2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}}{r^3}.$$

Since $\nabla \cdot \mathbf{B} = 0$, we know that $\nabla \cdot \mathbf{M} = 0$ and the magnetic scalar potential above satisfies Laplace's equation everywhere. Specifically,

$$\begin{aligned}
\psi_\mathrm{I} &= D\left(\frac{r}{b}\right)\cos\theta \\
\psi_\mathrm{II} &= \left[B\left(\frac{a}{r}\right)^2 + C\left(\frac{r}{a}\right)\right]\cos\theta \\
\psi_\mathrm{III} &= A\left(\frac{a}{r}\right)^2\cos\theta.
\end{aligned}$$

Applying the matching conditions, noting that $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ and that the only non-zero magnetization is

$$\mathbf{M}_\mathrm{II} = M\frac{2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}}{r^3},$$

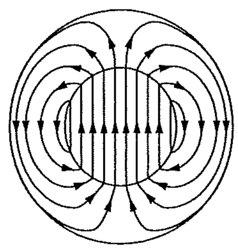
gives

$$\begin{aligned}
A &= B + C \\
D &= B\frac{a^2}{b^2} + C\frac{b}{a} \\
\frac{2M}{a^3} &= -\frac{2B}{a} + \frac{C}{a} + \frac{2A}{a} \\
-\frac{2M}{b^3} &= \frac{D}{b} + 2B\frac{a^2}{b^3} - \frac{C}{a}.
\end{aligned}$$

It is straightforward to check that this system is solved by

$$A = 0 \qquad C = -B = \frac{2M}{3a^2} \qquad D = B\left[\frac{a^2}{b^2} - \frac{b}{a}\right],$$

which confirms that $\mathbf{H} = \mathbf{B} = 0$ in region III outside the Moon. We can sketch $\mathbf{B}$ inside the Moon using the fact that $\mathbf{B}_\text{I} = \mathbf{H}_\text{I} = -\nabla\psi_\text{I}$ is constant, the lines of $\mathbf{B}$ must form closed loops, and $\mathbf{B}$ must be tangent to the sphere at $r = b$ because its radial component is continuous there.



Source: S.K. Runcorn, *Physics of the Earth and Planetary Interiors* **10**, 327 (1975).

13.13 Magnetic Shielding

We use a magnetic scalar potential where $\mathbf{H} = -\nabla\psi$. There is no free current, and the problem is two-dimensional, so

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(r\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} = 0.$$

By standard separation of variables, the general solution is a superposition of terms of the form

$$\psi(\rho,\theta) = (A_n \cos n\phi + B_n \sin n\phi)(C_n\rho^n + D_n\rho^{-n}).$$

Inside the shell, the solution must be finite and reflect the symmetry of the external field. Since $B_{\text{ext}} = \mu_0 H_{\text{ext}}$ and $\psi_{\text{ext}} = -H_{\text{ext}}x = -H_{\text{ext}}\rho\cos\phi$,

$$\psi_{\text{in}} = A\rho\cos\phi.$$

Within the shell, we have the slightly more general potential

$$\psi_{\text{shell}} = (C\rho + D\rho^{-1})\cos\phi.$$

Outside the shell, the field must reduce to $\mathbf{B}_{\text{ext}}$ as $\rho \to \infty$. Therefore,

$$\psi_{\text{out}} = -H_{\text{ext}}\rho\cos\phi + E\rho^{-1}\cos\phi.$$

The matching conditions are continuity for the normal component of $\mathbf{B}$ and continuity for the tangential component of $\mathbf{H}$. The latter is equivalent to the continuity of ψ itself. Applying these at $\rho = a$ gives

$$\left(\frac{\partial\psi_{\text{in}}}{\partial\rho}\right)_{\rho=a} = \kappa_m\left(\frac{\partial\psi_{\text{shell}}}{\partial\rho}\right)_{\rho=a} \quad \text{and} \quad \psi_{\text{in}}\big|_{\rho=a} = \psi_{\text{shell}}\big|_{\rho=a}$$

or

$$A = \kappa_m\left(C - \frac{D}{a^2}\right) \quad \text{and} \quad Aa = Ca + \frac{D}{a}.$$

The matching conditions at $\rho = b$ are

$$\left(\frac{\partial\psi_{\text{out}}}{\partial\rho}\right)_{\rho=a} = \kappa_m\left(\frac{\partial\psi_{\text{shell}}}{\partial\rho}\right)_{\rho=b} \quad \text{and} \quad \psi_{\text{out}}\big|_{\rho=a} = \psi_{\text{shell}}\big|_{\rho=b}$$

or

$$-H_{\text{ext}} = \frac{E}{b^2} = \kappa_m\left(C - \frac{D}{b^2}\right) \quad \text{and} \quad -H_{\text{ext}} + \frac{E}{b} = Cb + \frac{D}{b}.$$

From the matching conditions at $\rho = a$, we deduce that

$$\frac{C}{D} = \frac{\kappa_m + 1}{\kappa_m - 1}\frac{1}{a^2} \quad \Rightarrow \quad \frac{A}{D} = \frac{\kappa_m}{\kappa_m + 1}\frac{2}{a^2} \quad \Rightarrow \quad \frac{A}{C} = \frac{2\kappa_m}{\kappa_m + 1}. \tag{1}$$

Eliminating E from the matching conditions at $\rho = b$ gives

$$D = \frac{2H_{\text{ext}} + (\kappa_m + 1)C}{\kappa_m - 1} b^2.$$

Substituting this into the expression for C/D in (1) gives

$$C\left\{1 - \frac{b^2}{a^2}\left(\frac{\kappa_m + 1}{\kappa_m - 1}\right)^2\right\} = 2H_{\text{ext}}\frac{b^2}{a^2}\frac{\kappa_m + 1}{(\kappa_m - 1)^2}.$$

Using this to eliminate C from the expression for A/C in (1) gives

$$A = \frac{4\kappa_m b^2}{(\kappa_m - 1)^2 a^2 - (\kappa_m + 1)^2 b^2} H_{\text{ext}}.$$

This gives the advertised result because

$$\mathbf{B}_{\text{in}} = -\mu_0 A\hat{\mathbf{z}} = \frac{4\kappa_m b^2}{(\kappa_m + 1)^2 b^2 - (\kappa_m - 1)^2 a^2}\mathbf{B}_{\text{ext}}.$$