HW #2 Solutions

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12.2 A Hall Thruster

(a) We get uniform electron drift in the z-direction if the electric force density $-en_e\mathbf{E}$ exactly cancels the Lorentz magnetic force density $-en_e\mathbf{v}\times\mathbf{B}$. Imposing this condition, $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$, implies that

$$\mathbf{E} \times \mathbf{B} = \mathbf{B} \times (\mathbf{v} \times \mathbf{B}) = \mathbf{v}B^2 - \mathbf{B}(\mathbf{B} \cdot \mathbf{v}).$$

This, in turn, implies the suggested result,

$$\mathbf{v} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}.$$

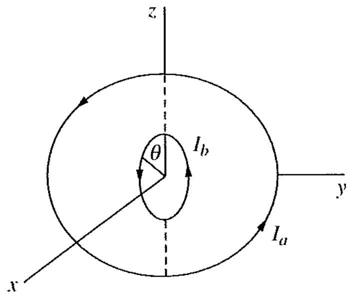
(b) Using the equality of the forces in part (a) and $n_i = n_e$, the electric force on the ions is

$$\mathbf{F}_i = en_i\mathbf{E} = en_e\mathbf{E} = -en_e\mathbf{v} \times \mathbf{B} = \mathbf{j}_{\rm Hall} \times \mathbf{B}.$$

By Newton's third law, the reaction thrust on the shells is $\mathbf{T} = -\mathbf{F}_i = \mathbf{B} \times \mathbf{j}_{\rm Hall}$ *if* the ions are ejected from V before the magnetic Lorentz force on the ions begins to act. This will be the case because a xenon ion is much more massive than an electron.

12.6 The Torque between Nested Current Rings

We locate the ring I_a in the x-y plane and the ring I_b in the x-z plane as shown below. The magnetic field on the axis of I_a points in the z-direction. The magnetic field of I_b points in the y-direction.



The vector torque which acts on I_b is

$$\mathbf{N} = I_b \oint \mathbf{r} \times (d\boldsymbol{\ell} \times \mathbf{B}),$$

where $\mathbf{r} = b\cos\theta\hat{\mathbf{z}} + b\sin\theta\hat{\mathbf{z}}$, $d\boldsymbol{\ell} = bd\theta\hat{\boldsymbol{\theta}} = bd\theta(\cos\theta\hat{\mathbf{x}} - \sin\theta\hat{\mathbf{z}})$, and $\mathbf{B} = B_\rho\hat{\boldsymbol{\rho}} + B_z\hat{\mathbf{z}}$ is the magnetic field due to I_a in cylindrical coordinates. Since parallel currents attract and anti-parallel currents repel, the only component of the torque which survives is N_x. Focusing on this component, substituting $\mathbf{r}$, $d\boldsymbol{\ell}$, and $\mathbf{B}$ into the torque formula gives

$$N_x = I_b b^2 \int_0^{2\pi} d\theta \cos\theta (B_z \cos\theta + B_\rho \sin\theta).$$

Our task now is to write the components of $\mathbf{B}(\rho, z)$ near the origin, where $\rho = b\sin\theta$ and $z = b\cos\theta$. In the text, we used the technique of "going off the axis" to find the exact magnetic scalar potential of a current loop. In polar coordinates, the first two terms of this expansion for $r < a$ were

$$\psi(r,\theta) = -\frac{1}{2}\mu_0 I_a \left[\frac{r}{a} P_0(0) P_1(\cos\theta) + \left(\frac{r}{a}\right)^3 P_2(0) P_3(\cos\theta)\right].$$

Using $P_0 = 1$, $P_1 = \cos\theta$, $P_2 = (1/2)(3\cos^2\theta - 1)$, and $P_3 = (1/2)(5\cos^3\theta - 3\cos\theta)$, together with $z = r\cos\theta$ and $\rho = r\sin\theta$, we get

$$\psi(\rho, z) = -\frac{1}{2}\mu_0 I_a \left[\frac{z}{a} - \frac{1}{2}\frac{z^3}{a^3} + \frac{3}{4}\frac{z\rho^2}{a^3}\right].$$

Therefore,

$$B_z = -\frac{\partial\psi}{\partial z} = \frac{\mu_0 I_a}{2a}\left[1 - \frac{3}{2}\frac{z^2}{a^2} + \frac{3}{4}\frac{\rho^2}{a^2}\right] = \frac{\mu_0 I_a}{2a}\left[1 - \frac{3}{2}\frac{b^2\cos^2\theta}{a^2} + \frac{3}{4}\frac{b^2\sin^2\theta}{a^2}\right]$$

and

$$B_\rho = -\frac{\partial\psi}{\partial\rho} = \frac{3\mu_0 I_a}{4}\frac{z\rho}{a^3} = \frac{3\mu_0 I_a}{4}\frac{b^2}{a^3}\sin\theta\cos\theta.$$

Substituting these fields into the torque formula and collecting terms gives the advertised result:

$$N_x = \frac{1}{2}\mu_0 I_a I_b \frac{b^2}{a}\int_0^{2\pi} d\theta \left[\cos^2\theta\left(1 - \frac{3b^2}{2a^2}\right) + \frac{15b^2}{16a^2}\sin^2 2\theta\right] = \frac{\pi}{2}\mu_0 I_a I_b \frac{b^2}{a}\left[1 - \left(\frac{3b}{4a}\right)^2\right].$$

12.19 Equivalence of Force Formulae

U_B must be expressed as a function of the flux variables. $\hat{U}_B$ must be expressed as a function of the current variables. To do this, we use

$$\Phi_k = M_{k\ell} I_\ell \qquad \text{and} \qquad I_k = M^{-1}_{k\ell}\Phi_\ell. \tag{1}$$

Therefore,

$$U_B = \frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k = \frac{1}{2}\sum_{k=1}^{N} \Phi_k M^{-1}_{k\ell}\Phi_\ell$$

$$\hat{U}_B = -\frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k = -\frac{1}{2}\sum_{k=1}^{N} I_k M_{k\ell} I_\ell.$$

Substituting these expressions into the force formulae in the statement of the problem shows that the proposition will be proved if we can show that

$$\Phi_k \nabla M^{-1}_{k\ell} \Phi_\ell = -I_k \nabla M_{k\ell} I_\ell. \tag{2}$$

We begin with $\mathbf{MM}^{-1} = \mathbf{I}$ written in component form:

$$M_{k\ell} M^{-1}_{\ell p} = \delta_{kp}.$$

Using this,

$$(\nabla M_{k\ell}) M^{-1}_{\ell p} + M_{k\ell}(\nabla M^{-1}_{\ell p}) = 0.$$

Multiplying on the right by M_{ps} and summing over p gives

$$(\nabla M_{k\ell}) M^{-1}_{\ell p} M_{ps} = -M_{k\ell}(\nabla M^{-1}_{\ell p}) M_{ps}.$$

Using the definition of the inverse,

$$(\nabla M_{k\ell}) \delta_{\ell s} = -M_{k\ell}(\nabla M^{-1}_{\ell p}) M_{ps}.$$

The left side of this equation is ∇M_{ks}. Therefore, using (1) and the fact that $M_{k\ell} = M_{\ell k}$,

$$\begin{aligned}
-I_k \nabla M_{ks} I_s &= -I_k \left[-M_{k\ell}(\nabla M^{-1}_{\ell p}) M_{ps} \right] I_s \\
&= I_k M_{k\ell} \nabla M^{-1}_{\ell p} \Phi_p \\
&= M_{\ell k} I_k \nabla M^{-1}_{\ell p} \Phi_p \\
&= \Phi_\ell \nabla M^{-1}_{\ell p} \Phi_p.
\end{aligned}$$

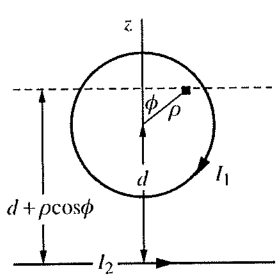
This is (2), as required.

12.20 The Force between a Current Loop and a Wire

(a) Let Φ_1 be the flux through the loop produced by the wire. The force on the loop is

$$\mathbf{F} = -\frac{\partial \hat{V}_B}{\partial d} \hat{\mathbf{z}},$$

where $\hat{V}_B$ is the interaction potential energy $\hat{V}_B = -I_1 \Phi_1$. The black square in the diagram below is an area element $dS = \rho d\rho d\phi$ at a distance $d + \rho \cos\phi$ from the wire.



The magnetic field at that element points out of the paper in the plane of the loop with magnitude

$$B_2 = \frac{\mu_0 I_2}{2\pi}\frac{1}{d+\rho\cos\phi}.$$

By the right-hand rule, $d\mathbf{S}$ points into the paper. Therefore,

$$\begin{aligned}\Phi_1 &= \int d\mathbf{S}\cdot\mathbf{B}_2 = -\frac{\mu_0 I_2}{2\pi}\int_0^R\int_0^{2\pi}\frac{\rho d\rho d\phi}{d+\rho\cos\phi} = -\mu_0 I_2\int_0^R \frac{\rho d\rho}{\sqrt{d^2-\rho^2}} \\ &= -\mu_0 I_2\left[\sqrt{d^2-R^2}-d\right].\end{aligned}$$

This gives the force on the loop as

$$\mathbf{F} = \mu_0 I_1 I_2 \frac{\partial}{\partial d}\left[\sqrt{d^2-R^2}-d\right]\hat{\mathbf{z}} = \mu_0 I_1 I_2\left[\frac{d}{\sqrt{d^2-R^2}}-1\right]\hat{\mathbf{z}}.$$

(b) In the limit $d \gg R$, we use $(1-R^2/d^2)^{-1/2} \approx 1 + R^2/2d^2$ to get the repulsive force on the loop as

$$\mathbf{F} = \frac{\mu_0 I_1 I_2 R^2}{2d^2}\hat{\mathbf{z}}.$$

The magnetic moment of the loop is $m_1 = I_1\pi R^2$ and points into the paper. Therefore, since $\mathbf{B}_2$ points out of the paper, the force on the loop should be

$$\mathbf{F} = \nabla(\mathbf{m}_1\cdot\mathbf{B}_2) = -(I_1\pi R^2)\frac{\mu_0 I_2}{2\pi}\frac{\partial}{\partial d}\left(\frac{1}{d}\right)\hat{\mathbf{z}} = \frac{\mu_0 I_1 I_2 R^2}{2d^2}\hat{\mathbf{z}}.$$