HW #1 solutions

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11.1 Magnetic Dipole Moment Practice

We will find the current using $\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{j}$. First,

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0 A_0}{4\pi}\left[\hat{\mathbf{r}}\frac{2\cos\theta}{r^2} + \hat{\boldsymbol{\theta}}\frac{\lambda\sin\theta}{r}\right]\exp(-\lambda r).$$

Therefore,

$$\mathbf{j} = \frac{1}{\mu_0}\nabla \times \mathbf{B} = \hat{\boldsymbol{\phi}} 4\pi A_0 \sin\theta\left\{\frac{2}{r^3} - \frac{\lambda^2}{r}\right\}\exp(-\lambda r).$$

The associated magnetic moment is

$$\mathbf{m} = \tfrac{1}{2}\int d^3r\, \mathbf{r} \times \mathbf{j} = -\frac{A_0}{8\pi}\int d^3r\, \hat{\boldsymbol{\theta}}\, r\sin\theta\left\{\frac{2}{r^3} - \frac{\lambda^2}{r}\right\}\exp(-\lambda r).$$

But $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$. This shows that only the $\hat{\mathbf{z}}$-component survives the integration. Hence,

$$\begin{aligned}
\mathbf{m} &= \hat{\mathbf{z}}\frac{A_0}{4}\int_0^\pi d\theta\, \sin^2\theta \int_0^\infty dr r^3 \exp(-\lambda r)\left\{\frac{2}{r^3} - \frac{\lambda^2}{r}\right\} \\
&= \hat{\mathbf{z}}\frac{\pi A_0}{8}\left\{2 - \lambda^2\frac{d^2}{d\lambda^2}\right\}\int_0^\infty dr \exp(-\lambda r) \\
&= 0.
\end{aligned}$$

11.4 The Magnetic Moment of a Rotating Charged Disk

Orient the disk to lie in the x-y plane with its center at the origin and let $\boldsymbol{\rho}$ be the radial vector (in polar coordinates) in that plane. If each element of surface charge $dq = \sigma dS$ moves with velocity $\mathbf{v} = \boldsymbol{\omega} \times \boldsymbol{\rho}$, the magnetic dipole moment of the disk is

$$\mathbf{m} = \frac{1}{2}\int \boldsymbol{\rho} \times dq\mathbf{v} = \frac{1}{2}\sigma \int dS \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}). \tag{11.2}$$

(a) The rotation axis is normal to the plane. In other words, $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$ is perpendicular to $\boldsymbol{\rho}$. Therefore,

$$\mathbf{m} = \frac{1}{2}\sigma \int dS \left[\boldsymbol{\omega}\rho^2 - \boldsymbol{\rho}(\boldsymbol{\omega} \cdot \boldsymbol{\rho})\right] = \frac{1}{2}\sigma \int_0^{2\pi} d\phi \int_0^R d\rho\rho^3 \boldsymbol{\omega} = \frac{\pi}{4}\sigma R^4 \boldsymbol{\omega}.$$

(b) We can choose $\boldsymbol{\omega} = \omega\hat{\mathbf{x}}$ since the rotation axis lies along a diameter. Then, because $\boldsymbol{\rho} = \rho\cos\phi\hat{\mathbf{x}} + \rho\sin\phi\hat{\mathbf{y}}$, (11.2) becomes

$$\mathbf{m} = \frac{1}{2}\sigma\omega \int dS \left[\rho^2 \sin^2\phi\hat{\mathbf{x}} - \rho^2 \sin\phi\cos\phi\hat{\mathbf{y}}\right] = \frac{\pi}{8}\sigma R^4 \boldsymbol{\omega}.$$

11.9 Magnetic Dipole and Quadrupole Moments for $\psi(\mathbf{r})$

(a) We get a Cartesian expansion using

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} - \mathbf{r}'\cdot\nabla\frac{1}{r} + \frac{1}{2}(\mathbf{r}'\cdot\nabla)^2\frac{1}{r} - \cdots .$$

Inserting this above gives

$$I = \frac{\mu_0}{4\pi}\left[\frac{1}{r}\int d^3r'\,\mathbf{r}'\cdot\nabla'\times\mathbf{j}\,(\mathbf{r}') - \int d^3r'\,\mathbf{r}'\cdot\nabla'\times\mathbf{j}\,(\mathbf{r}')\mathbf{r}'\cdot\nabla\frac{1}{r} + \int d^3r'\,\mathbf{r}'\cdot\nabla'\times\mathbf{j}\,(\mathbf{r}')\frac{1}{2}(\mathbf{r}'\cdot\nabla)^2\frac{1}{r}\right].$$

The first integral in the square brackets vanishes because the identity

$$\nabla\cdot(\mathbf{r}\times\mathbf{j}) = \mathbf{j}\cdot(\nabla\times\mathbf{r}) - \mathbf{r}\cdot\nabla\times\mathbf{j} = -\mathbf{r}\cdot\nabla\times\mathbf{j} \qquad (1)$$

and Gauss' law produce a surface integral at infinity which is zero for a localized current distribution. The two terms which remain are exactly

$$I = -2\mathbf{m}\cdot\nabla\frac{1}{r} + m_{ij}^{(2)}\nabla_i\nabla_j\frac{1}{r}.$$

Now, $\nabla_i(1/r) = f(\theta,\phi)/r^2$ and $\nabla_i\nabla_j(1/r) = g(\theta,\phi)/r^3$. Therefore, since $I = -r\partial\psi/\partial r$, we get the advertised expansion for $\psi(\mathbf{r})$ immediately.

(b) Using (1) and integrating by parts,

$$\begin{aligned}\mathbf{m} &= \frac{1}{2}\int d^3r\,(\mathbf{r}\cdot\nabla\times\mathbf{j})\mathbf{r}\\ &= -\frac{1}{2}\int d^3r\,\nabla\cdot(\mathbf{r}\times\mathbf{j})\mathbf{r}\\ &= \frac{1}{2}\int d^3r\,(\mathbf{r}\times\mathbf{j})\cdot\nabla\mathbf{r}\\ &= \frac{1}{2}\int d^3r\,(\mathbf{r}\times\mathbf{j}).\end{aligned}$$

(c) Again using (1) and integrating by parts,

$$\begin{aligned}m_{ij}^{(2)} &= \frac{1}{2}\int d^3r\,(\mathbf{r}\cdot\nabla\times\mathbf{j})r_ir_j\\ &= -\frac{1}{2}\int d^3r\,\nabla\cdot(\mathbf{r}\times\mathbf{j})r_ir_j\\ &= \frac{1}{2}\int d^3r\,(\mathbf{r}\times\mathbf{j})\cdot\nabla(r_ir_j).\end{aligned}$$

(d) Because $\mathbf{r}\times\mathbf{j}$ is perpendicular to $\mathbf{r}$,

$$\mathrm{Tr}\,\mathbf{m}^{(2)} = \sum_{i=1}^{3} m_{ii}^{(2)} = \frac{1}{2}\int d^3r\,(\mathbf{r}\times\mathbf{j})\cdot\nabla(r^2) = \int d^3r\,(\mathbf{r}\times\mathbf{j})\cdot\mathbf{r} = 0.$$

(e) Using the result of part (c),

$$\begin{aligned} m_{ij}^{(2)} &= \frac{1}{2}\int d^3r\,(\mathbf{r}\times\mathbf{j})_k \nabla_k (r_i r_j) \\ &= \frac{1}{2}\int d^3r\,(\mathbf{r}\times\mathbf{j})_k (\delta_{ki} r_j + r_i \delta_{kj}) \\ &= \frac{1}{2}\int d^3r\,[(\mathbf{r}\times\mathbf{j})_i r_j + r_i (\mathbf{r}\times\mathbf{j})_j]. \end{aligned}$$

(f) The key point is that $m_{ij}^{(2)} = \frac{1}{2}(M_{ij}^{(2)} + M_{ji}^{(2)})$. Therefore,

$$\begin{aligned} m_{ij}^{(2)}\nabla_i\nabla_j\frac{1}{r} &= \frac{1}{2}\left[M_{ij}^{(2)}\nabla_i\nabla_j\frac{1}{r} + M_{ji}^{(2)}\nabla_i\nabla_j\frac{1}{r}\right] \\ &= \frac{1}{2}\left[M_{ij}^{(2)}\nabla_i\nabla_j\frac{1}{r} + M_{ij}^{(2)}\nabla_j\nabla_i\frac{1}{r}\right] = M_{ij}^{(2)}\nabla_i\nabla_j\frac{1}{r}. \end{aligned}$$

Source: C.G. Gray, *American Journal of Physics* **46**, 582 (1978); *ibid.* **48**, 984 (1980).