Name:

Final E&M 504, Spring 2017

Problem 1 [10 points]

Consider a planar spiral composed of N turns of a wire which carries a steady current I. If the inner radius of the spiral is a and the outer radius of the spiral is b, what is its total magnetic moment?

One turn of the wire with radius r produces magnetic moment $m(r) = I\pi r^2$. In our system, the number of turns per unit radius is uniform: $\frac{dN}{dr} = \frac{N}{b-a}$ $(b>a)$

Thus the total magnetic moment is $m_{tot} = \int_a^b m(r) \frac{dN}{dr} dr =$

$$= \frac{I\pi N}{b-a} \int_a^b dr\, r^2 = \frac{I\pi N}{b-a} \frac{b^3 - a^3}{3}.$$

The direction of $\vec{m}_{tot}$ depends on the direction of I and is given by the right-hand rule.

Problem 2 [10 points]

Consider a gauge transformation:

$$(*) \quad \begin{cases} \vec{A}' = \vec{A} + \vec{\nabla}\Lambda, \\ \varphi' = \varphi - \dfrac{\partial \Lambda}{\partial t} \end{cases}, \quad \text{where}$$

$\Lambda(\vec{r},t)$ is an arbitrary gauge function.

(a) Show that $\vec{E}$ & $\vec{B}$ are invariant under $(*)$.

(b) Derive wave equations for φ and $\vec{A}$ using the Lorenz gauge.

(c) If $\varphi, \vec{A}$ are in the Lorenz gauge, what equation must Λ satisfy so that $\varphi', \vec{A}'$ are also in the Lorenz gauge? What happens to the wave equations from (b) if this equation for Λ is obeyed?

(a) Under (*),

$$\vec{B}' = \vec{\nabla} \times \vec{A}' = \underbrace{\vec{\nabla} \times \vec{A}}_{\vec{B}} + \underbrace{\vec{\nabla} \times \vec{\nabla} \Lambda}_{0} = \vec{B},$$

$$\vec{E}' = -\vec{\nabla} \varphi' - \frac{\partial \vec{A}'}{\partial t} = \vec{E} + \frac{\partial}{\partial t} \vec{\nabla} \Lambda -$$

$$- \frac{\partial}{\partial t} \vec{\nabla} \Lambda = \vec{E}$$

(b) The two inhomogeneous Maxwell eq's become:

$$\begin{cases} \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \varphi}{\partial t} \right) = -\mu_0 \vec{J} \\ \nabla^2 \varphi + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) = -\frac{\rho}{\epsilon_0} \end{cases}$$

In the Lorentz gauge,

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \varphi}{\partial t} = 0,$$ so that

$$(**) \quad \begin{cases} \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}, \\ \nabla^2 \varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \end{cases}$$ become indep. wave eq's

(c) $$\vec{\nabla} \cdot \vec{A}' + \frac{1}{c^2} \frac{\partial \varphi'}{\partial t} = \vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \varphi}{\partial t} +$$

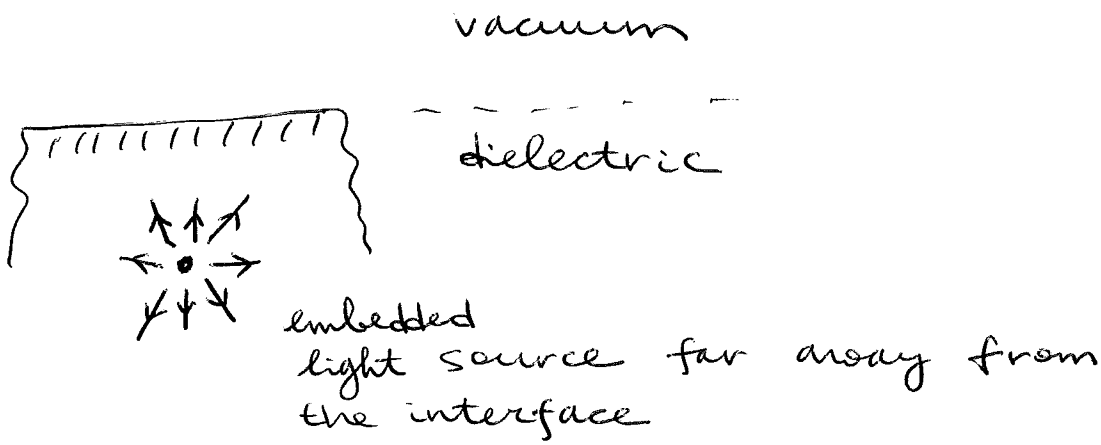
$$+ \nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2}.$$

Thus $\vec{A}', \varphi'$ will obey the Lorentz gauge if $\underline{\underline{\nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} = 0}}$

Clearly, the wave eq's in (**) will remain invariant if Λ satisfies the (homogeneous) eq'n above.

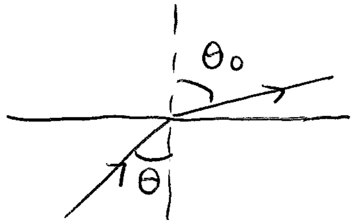
Problem 3 [10 points]

A point light source is embedded inside a semi-infinite dielectric with index of refraction n. The source is sufficiently far from the dielectric-vacuum interface, so that its light can be treated as plane waves (light rays). What fraction of rays escapes the dielectric? What happens in the $n \to 1$ limit?



We can use geometric optics in this problem. Snell's law:

$$n \sin\theta = \sin\theta_0$$



The critical angle is $\theta_0^c = \frac{\pi}{2} \Rightarrow \sin\theta_0^c = 1$, and correspondingly

$$\theta_c = \sin^{-1}\frac{1}{n}.$$

The fraction of rays that escape (i.e. do not undergo total internal reflection) is then

$$F = \underbrace{\frac{1}{4\pi}}_{\text{total solid angle}} \int_0^{2\pi} d\varphi \int_0^{\theta_c} d\theta \sin\theta = \frac{1}{2}(1-\cos\theta_c) =$$

$$= \frac{1}{2}\left(1-\sqrt{1-\frac{1}{n^2}}\right).$$

Note that if $n=1$ (no dielectric), $F=\frac{1}{2}$, as expected.

Half of the rays will cross the (imaginary) boundary.

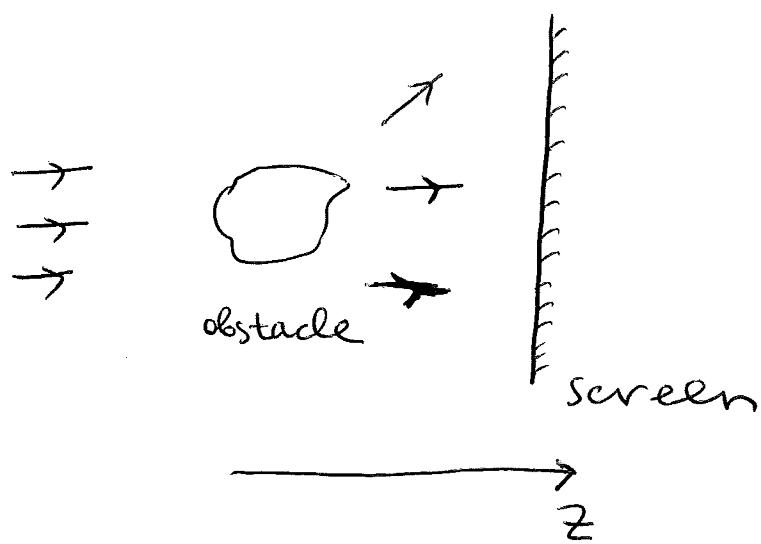
Problem 4 [10 points]

A unit-amplitude, monochromatic scalar plane wave scatters from an origin-centered obstacle of finite size:

$$\psi(r,\theta,\varphi) \approx e^{ikz} + \frac{e^{ikr}}{r} f(\theta,\varphi).$$

Show that $\sigma_{tot} = \sigma_{sc} + \sigma_{abs} = \frac{4\pi}{k} \operatorname{Im} \underbrace{f(0,0)}_{\text{forward scattering amplitude}}$.

Hint: consider a flat screen far behind the obstacle and evaluate $\int_{screen} dS\, |\psi|^2$, where dS is the element of the screen's area.



The scattering will be mostly in the forward direction. If the screen is placed far behind the obstacle, we can assume $x, y \ll z$ and thus

$$r = z\sqrt{1 + \frac{x^2+y^2}{z^2}} \simeq z + \frac{\rho^2}{2z} \quad , \text{ where}$$

$$\rho^2 = x^2 + y^2 .$$

Hence $$\psi(\vec{r}) \simeq e^{ikz} + \frac{e^{ikz}}{z} e^{i\frac{k\rho^2}{2z}} \underbrace{f(0,0)}_{\substack{\text{forward}\\ \text{scattering}\\ \text{amplitude}}}$$

Then $$|\psi|^2 \simeq 1 + \frac{2}{z} \mathrm{Re}\left\{ f(0,0)\, e^{i\frac{k\rho^2}{2z}} \right\} .$$

Integrating over the screen, we obtain:

$$\int_{\text{screen}} ds\, |\psi|^2 \simeq \int_{\text{screen}} ds + \frac{2}{z} \mathrm{Re}\left\{ \underbrace{f(0,0)}_{\mathrm{Re}+i\mathrm{Im}} \int_{\text{screen}} ds\, e^{i\frac{k\rho^2}{2z}} \right\} =$$

$$\int_{-\infty}^{\infty} dx\, e^{i\frac{kx^2}{2z}} \int_{-\infty}^{\infty} dy\, e^{i\frac{ky^2}{2z}} =$$

$$= \frac{1}{\left(i\sqrt{\frac{ik}{2z}}\right)^2} \Big(\underbrace{\int_{-\infty}^{\infty} du\, e^{-u^2}}_{\sqrt{\pi}} \Big)^2 = \frac{2\pi i z}{k} .$$

$$= \underbrace{\int_{\text{screen}} ds}_{\text{cross-section without the obstacle}} - \underbrace{\frac{4\pi}{k}\,\mathrm{Im}\, f(0,0)}_{\text{scattering and absorption losses}} .$$

Therefore,

$$\sigma_{tot} = \frac{4\pi}{k}\,\mathrm{Im}\, f(0,0) .$$

Problem 5 [20 points]

(a) For two 4-vectors $A = (A_0, \vec{A})$ and $B = (B_0, \vec{B})$, show explicitly that the scalar product $AB = A_0 B_0 - \vec{A}\cdot\vec{B}$ is a Lorentz invariant scalar.

Hint: write $\vec{A} = \vec{A}_{\parallel} + \vec{A}_{\perp}$ and $\vec{B} = \vec{B}_{\parallel} + \vec{B}_{\perp}$, where $\parallel$ & $\perp$ directions are wrt $\vec{v}$ which defines the Lorentz boost.

(b) A point particle with charge q and mass m is moving along the y-axis. The initial energy, momentum, and velocity are $\mathcal{E}_0$, $\vec{p}_0 = p_0 \hat{y}$, and $\vec{u}_0 = u_0 \hat{y}$, respectively. At $t=0$, $\vec{E} = E\hat{z}$ switches on. Find the relativistic EoM for the particle, both in terms of $y(t)$, $z(t)$ & $z(y)$. Sketch the trajectory and check the non-relativistic limit.

(a) Recall that if $\vec{\beta} = \frac{\vec{v}}{c}$, the Lorentz transformation is given by

$$\begin{cases} A_0' = \gamma (A_0 - \vec{\beta}\cdot\vec{A}), \\ A_\parallel' = \gamma (A_\parallel - \beta A_0), \\ \vec{A}_\perp' = \vec{A}_\perp, \end{cases} \qquad \vec{A} = A_\parallel \vec{e}_\parallel + \vec{A}_\perp$$

and similarly for B.

Therefore,

$$A_0' B_0' - \vec{A}'\cdot\vec{B}' = \gamma^2 (A_0 - \vec{\beta}\cdot\vec{A})(B_0 - \vec{\beta}\cdot\vec{B}) -$$

$$- (A_\parallel' \vec{e}_\parallel + \vec{A}_\perp')(B_\parallel' \vec{e}_\parallel + \vec{B}_\perp') =$$

$$= \gamma^2 A_0 B_0 - \gamma^2 (\vec{\beta}\cdot\vec{A}) B_0 - \gamma^2 (\vec{\beta}\cdot\vec{B}) A_0 +$$

$$+ \gamma^2 (\vec{\beta}\cdot\vec{A})(\vec{\beta}\cdot\vec{B}) - \gamma^2 (A_\parallel - \beta A_0)(B_\parallel - \beta B_0) -$$

$$- \vec{A}_\perp\cdot\vec{B}_\perp = \underbrace{\gamma^2 (1-\beta^2)}_{1} A_0 B_0 - \gamma^2 \cancel{\beta A_\parallel B_0} -$$

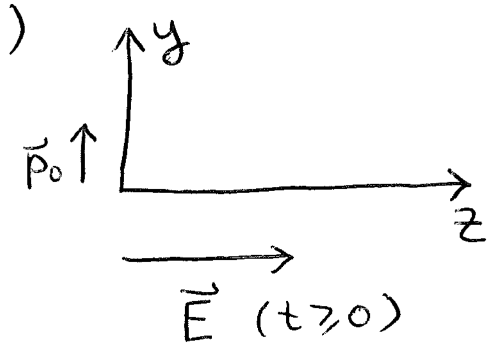
$$- \gamma^2 \cancel{\beta B_\parallel A_0} - \gamma^2 A_\parallel B_\parallel + \gamma^2 \cancel{\beta A_0 B_\parallel} +$$

$$+ \gamma^2 \cancel{\beta B_0 A_\parallel} - \vec{A}_\perp\cdot\vec{B}_\perp + \gamma^2 \beta^2 A_\parallel B_\parallel =$$

$$= A_0 B_0 - \underbrace{\gamma^2 (1-\beta^2)}_{1} A_\parallel B_\parallel - \vec{A}_\perp\cdot\vec{B}_\perp =$$

$$= A_0 B_0 - \vec{A}\cdot\vec{B},$$ as desired

(b)



The EoM is:

$$\dot{\vec{p}} = q\vec{E},$$

where $\vec{p} = \gamma m \vec{u}$.

Thus $\vec{p} = q\vec{E}t + \vec{p}(0)$.

$$\begin{cases} p_z = qEt & (p_z(0) = 0) \\ p_y = p_0 \end{cases}$$

Recall that

$$\mathcal{E}^2 = p^2c^2 + (mc^2)^2 =$$
$$= \left[(qEt)^2 + p_0^2\right]c^2 + (mc^2)^2 =$$
$$= \mathcal{E}_0^2 + c^2(qEt)^2,$$

where $\mathcal{E}$ is the particle's energy and $\mathcal{E}_0$ is the initial energy.

Then $\dfrac{dz}{dt} = u_z = \dfrac{p_z c^2}{\mathcal{E}} = \dfrac{qEtc^2}{\sqrt{\mathcal{E}_0^2 + (qEtc)^2}} =$

$\mathcal{E} = \gamma m c^2, \quad \vec{u} = \dfrac{\vec{p}}{\gamma m} = \dfrac{\vec{p}c^2}{\mathcal{E}}$

$$= \frac{1}{qE}\frac{d}{dt}\sqrt{\mathcal{E}_0^2 + (qEtc)^2}.$$

Note that $u_z \to c$ as $t \to +\infty$.

Thus $$z(t) = \frac{1}{qE}\sqrt{\mathcal{E}_0^2 + (qEtc)^2} + \text{const}. \quad (*)$$

$$z(t) \to ct \quad \text{as } t \to +\infty.$$

Likewise, $$u_y = \frac{dy}{dt} = \frac{p_0 c^2}{\mathcal{E}} = \frac{p_0 c^2}{\sqrt{\mathcal{E}_0^2 + (qEtc)^2}}.$$

As $t \to +\infty$, $u_y \to 0$ as $1/t$.

Next, $$y(t) = \frac{p_0 c}{qE} \underbrace{\operatorname{sh}^{-1} \frac{qEtc}{\mathcal{E}_0}}_{\text{“}\varphi\text{, s.t.}} + \text{const} \quad (**)$$

$$\operatorname{sh}\varphi = \frac{qEtc}{\mathcal{E}_0}.$$

Neglecting the constants in $(*)$ and $(**)$ for simplicity, we obtain:

$$\begin{cases} y = \dfrac{p_0 c}{qE}\varphi, \\ z = \dfrac{\mathcal{E}_0}{qE}\underbrace{\sqrt{1 + \left(\dfrac{qEtc}{\mathcal{E}_0}\right)^2}}_{\sqrt{1+\operatorname{sh}^2\varphi} = \operatorname{ch}\varphi} = \dfrac{\mathcal{E}_0}{qE}\operatorname{ch}\varphi. \end{cases}$$

These are the $y(t)$, $z(t)$ equations.

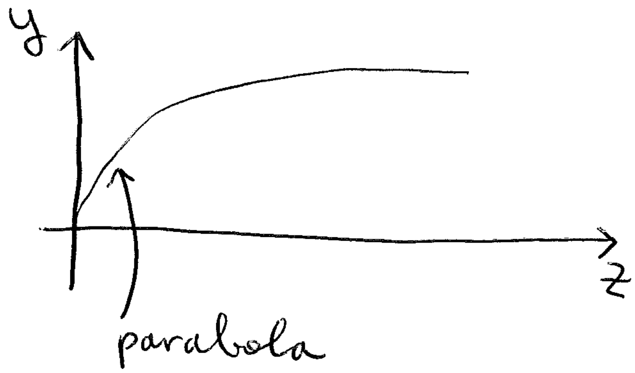
Finally, $$\underline{\underline{z = \frac{\mathcal{E}_0}{qE}\operatorname{ch}\left[\frac{qE}{p_0 c} y\right]}}.$$

In the non-relativistic limit,

$qEtc \ll \mathcal{E}_0$, yielding $\varphi \ll 1$

and $\operatorname{ch}\varphi \simeq 1 + \frac{\varphi^2}{2}$

Thus $z \simeq \frac{\mathcal{E}_0}{qE}\left[1 + \left(\frac{y\,qE}{p_0 c}\right)^2\right]$ is a parabola. The trajectory looks like:



Asymptotically, the particle moves along the z-axis with velocity c, while its y-coordinate does not change anymore.