Final Exam:
E&M II 504

Spring 2016

Problem 1 [10 points]

Consider a stationary particle with charge e. An incoming plane wave with electric field $\vec{E} = \vec{E}_0 e^{-i\omega t}$ (where $\vec{E}_0 = E_0\vec{\epsilon}_0$) interacts with the particle, inducing a time-dependent dipole moment.

For the *unpolarized* incident radiation, derive the differential scattering cross section (per unit solid angle, summed over scattered polarization), the polarization $\Pi(\theta)$, and the total scattering cross-section.

EoM: $m\ddot{\vec{x}} = -e\vec{E} = -e\vec{E}_0 e^{-i\omega t}$

$$\vec{x} = -\frac{e\vec{E}_0 e^{-i\omega t}}{m(i\omega)^2} = \frac{e\vec{E}_0 e^{-i\omega t}}{m\omega^2}$$

Dipole moment: $\vec{p} = -e\vec{x} = -\dfrac{e^2\vec{E}}{m\omega^2}$.

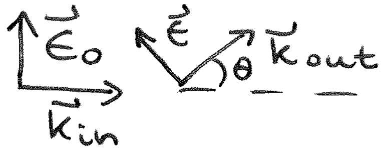
Electric dipole scattering:

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}|\vec{\epsilon}^* \cdot \vec{p}|^2 \underset{\omega = ck}{=} \underbrace{\left(\frac{e^2}{4\pi\epsilon_0 mc^2}\right)^2}_{\sigma_0} |\vec{\epsilon}^* \cdot \vec{\epsilon}_0|^2$$

Two cases: (1) $\vec{\epsilon}_0 \parallel$ scattering plane
(2) $\vec{\epsilon}_0 \perp$ scattering plane

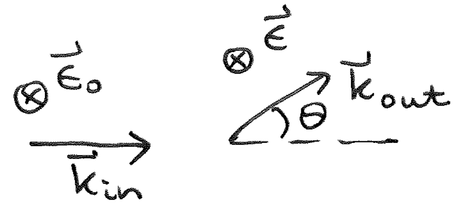
↑ defined by $\vec{k}_{in}$ & $\vec{k}_{out}$

(1) $\parallel$



$$\vec{\epsilon}_0 \cdot \vec{\epsilon} = \cos\theta$$

(2) $\perp$



$$\vec{\epsilon}_0 \cdot \vec{\epsilon} = 1$$

So, $\dfrac{d\sigma_{\parallel}}{d\Omega} = \sigma_0 \cos^2\theta \qquad \dfrac{d\sigma_{\perp}}{d\Omega} = \sigma_0$

Unpolarized light:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{d\sigma_{\parallel}}{d\Omega} + \frac{d\sigma_{\perp}}{d\Omega}\right) = \frac{\sigma_0}{2}(1+\cos^2\theta).$$

$$\Pi(\theta) = \frac{\frac{d\sigma_{\perp}}{d\Omega} - \frac{d\sigma_{\parallel}}{d\Omega}}{\frac{d\sigma_{\perp}}{d\Omega} + \frac{d\sigma_{\parallel}}{d\Omega}} = \frac{\sin^2\theta}{1+\cos^2\theta}$$

Finally, $\sigma = \int d\Omega \dfrac{d\sigma}{d\Omega} = \dfrac{8\pi}{3}\sigma_0 .$

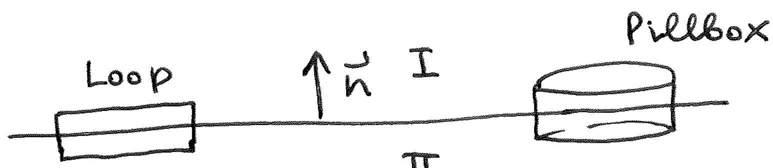
Problem 2 [10 points]

electrostatics

Consider an interface between two dielectric media, with no free surface charges.

a) Which components of the electric field $\vec{E}$ are continuous across the boundary surface? Prove it.

b) Which components of the displacement field $\vec{D}$ are continuous across the boundary surface? Prove it.



$\vec{\nabla}\times\vec{E}=0$ $\qquad$ $\vec{\nabla}\cdot\vec{D}=0$

$$\begin{cases} \oint d\vec{r}\cdot\vec{E} = \int d\vec{s}\cdot(\vec{\nabla}\times\vec{E}) = \\ = 0, & (1) \\ \int d\vec{s}\cdot\vec{D} = \int d^3r\,(\vec{\nabla}\cdot\vec{D}) = 0. & (2) \end{cases}$$

Shrink short sides to zero length:

(1) $\vec{E}$ $\parallel$ surface continuous:

$$\vec{n}\times\vec{E}_{I} = \vec{n}\times\vec{E}_{II}$$

(2) $\vec{D}$ $\perp$ surface continuous:

$$\vec{n}\cdot\vec{D}_{I} = \vec{n}\cdot\vec{D}_{II}$$

Problem 3 [10 points]

In free space, write down the expressions for the energy density u and the energy flux vector $\vec{S}$. Prove that

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0.$$

Gaussian units:

$$\begin{cases} u = \dfrac{1}{8\pi}(\vec{E}^2 + \vec{B}^2), \\ \vec{S} = \dfrac{c}{4\pi} \vec{E} \times \vec{B} \end{cases}$$

Then $$\frac{\partial u}{\partial t} = \frac{1}{4\pi}\Big(\vec{E} \cdot \underbrace{\frac{\partial \vec{E}}{\partial t}}_{c(\vec{\nabla}\times\vec{B})} + \vec{B} \cdot \underbrace{\frac{\partial \vec{B}}{\partial t}}_{-c(\vec{\nabla}\times\vec{E})}\Big) =$$

$$= \frac{c}{4\pi}\left(\vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \vec{B}\cdot(\vec{\nabla}\times\vec{E})\right).$$

Further, $$\vec{\nabla}\cdot\vec{S} = \vec{\nabla}\cdot\left[\frac{c}{4\pi}(\vec{E}\times\vec{B})\right] =$$

$$= \frac{c}{4\pi}\left[\vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B})\right].$$

Clearly, $$\frac{\partial u}{\partial t} = -\vec{\nabla}\cdot\vec{S}$$

Problem 4 [10 points]

Suppose that the scalar potential V is identically zero and the vector potential $\vec{A}$ is given by $(0, A_0 \sin(kx - \omega t), 0)$, where k and ω are constants. Compute the $\vec{E}$ and $\vec{B}$ fields. What relationship between k and ω is consistent with Maxwell's equations?

SI units:

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = \omega A_0 \cos(kx - \omega t)\, \hat{y}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{\partial}{\partial x}\left(A_0 \sin(kx - \omega t)\right) \hat{z} =$$

$$= A_0 k \cos(kx - \omega t)\, \hat{z}.$$

$$\vec{\nabla} \times \vec{B} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \quad \text{then yields}$$

$$-\frac{\partial}{\partial x}\left(A_0 k \cos(kx - \omega t)\right) \hat{y} = \mu_0 \epsilon_0 A_0 \omega^2 \times$$

$$\times \sin(kx - \omega t)\, \hat{y}, \text{ or}$$

$$k^2 = \underbrace{\mu_0 \epsilon_0}_{1/c^2} \omega^2 \quad \Rightarrow \quad \omega = ck.$$

Problem 5 [10 points]

Consider a particle of charge q moving in a circle of radius R at a constant angular velocity ω around the z axis. Find the Lienard-Wiechert potentials $\Phi(z,t)$ and $\vec{A}(z,t)$ on the z axis. Write them out explicitly in terms of ω, R, and the retarded time (which should also be shown explicitly).

Recall that in general

$$\begin{cases} \varphi(\vec{x},t) = \left.\dfrac{q}{(1-\vec{\beta}\cdot\vec{n})R'}\right|_{ret}, \\ \vec{A}(\vec{x},t) = \left.\dfrac{q\vec{\beta}}{(1-\vec{\beta}\cdot\vec{n})R'}\right|_{ret} \end{cases}$$

Here $\vec{\beta}\cdot\vec{n} = 0$ since $\vec{\beta} \perp \vec{r}$.

So $\varphi(z,t) = \left.\dfrac{q}{R'}\right|_{ret} = \dfrac{q}{\sqrt{z^2+R^2}}$.

Choose initial conditions s.t.

$\vec{x}_0(t) = (\cos\omega t, \sin\omega t, 0)$

Then $\vec{v}_0(t) = (-\omega\sin\omega t, \omega\cos\omega t, 0)$.

$$\vec{A}(z,t) = \left.\frac{q\vec{\beta}}{R'}\right|_{ret} = \frac{q\omega}{c\sqrt{z^2+R^2}}(-\sin\omega t_r, \cos\omega t_r, 0)$$

where $t_r = t - \dfrac{\sqrt{z^2+R^2}}{c}$.

Problem 6 [10 points]

Find the Green's function $G(\vec{r}-\vec{r}')$ of the Poisson equation:

$$\nabla^2\phi(\vec{r}) = -4\pi\rho(\vec{r}),$$

where $\phi(\vec{r})$ is the electrostatic potential and $\rho(\vec{r})$ is the charge density. For simplicity, first consider the point source at the origin:

$$\nabla^2 G(\vec{r}) = -4\pi\delta(\vec{r}),$$

and then generalize to $G(\vec{r}-\vec{r}')$.
What is the solution of the Poisson equation in terms of the Green's function?

$G(\vec{r})$ is the field due to a point source at the origin. Thus we expect that $G(\vec{r}) = G(r)$ due to spherical symmetry, and that $G(r) \to 0$ as $r \to +\infty$.

$$r \neq 0 \Rightarrow \nabla^2 G(r) = 0$$

$$\frac{\partial G}{\partial x_i} = \underbrace{G'}_{\frac{dG}{dr}} \frac{x_i}{r}$$

$$\sum_i \frac{\partial^2}{\partial x_i \partial x_i} G = G'' + \frac{2}{r} G'$$

So, $$\frac{G''}{G'} = -\frac{2}{r} \Rightarrow G'(r) = \frac{C_1}{r^2},$$

$$G(r) = -\frac{C_1}{r} + C_2.$$

$G(r) \to 0$ as $r \to +\infty$: $C_2 = 0$.

Now, we expect

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(\vec{r})$$

consistent with Coulomb's law (i.e. experimental observe.)

So, $C_1 = -1 \Rightarrow G(r) = \frac{1}{r}$.

Correspondingly, $G(\vec{r}-\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|}$.

Then $\varphi(\vec{r}) = \int d^3r'\, \rho(\vec{r}\,')\, G(\vec{r}-\vec{r}\,')$ is the solution of the Poisson equation. Indeed,

$$\nabla^2\varphi = \int d^3r'\, \rho(\vec{r}\,')\, \underbrace{\nabla^2 G(\vec{r}-\vec{r}\,')}_{-4\pi\delta(\vec{r}-\vec{r}\,')} =$$

$$= -4\pi\rho(\vec{r}),$$ as expected.