Final Exam:
E&M II 504

Spring 2015

Problem 1 [10 points]

The charge and current densities obey the conservation law:

$$\frac{\partial \rho(\vec{x},t)}{\partial t} + \vec{\nabla} \cdot \vec{j}(\vec{x},t) = 0.$$

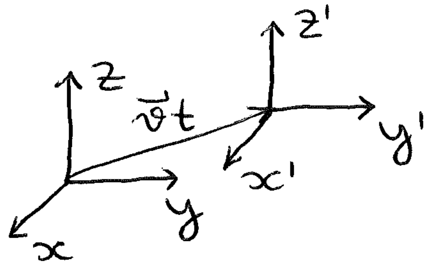
a) Explain how this law accounts for the rate of change of the charge ΔQ in some volume ΔV, in terms of the total current ΔI that flows through the surface of the volume.

$$\frac{d}{dt}\Delta Q = \frac{d}{dt}\int_{\Delta V} d^3x\, \rho(\vec{x},t) = \int_{\Delta V} d^3x\, \frac{\partial \rho}{\partial t} =$$

$$= -\int_{\Delta V} d^3x\, \vec{\nabla}\cdot\vec{j} = -\int_{\underset{\substack{\uparrow \\ \text{surface of } \Delta V}}{\Delta S}} d\vec{S}\cdot\vec{j} = \underline{\underline{-\Delta I}}\,.$$

Thus $\frac{d}{dt}\Delta Q = -\Delta I$, as expected

b) Consider the Galilean transformation to a new frame moving with the velocity $\vec{v}$. How are the charge and current densities $\rho'(\vec{x}', t')$ and $\vec{j}'(\vec{x}', t')$ in the new frame related to the charge and current densities $\rho(\vec{x}, t)$ and $\vec{j}(\vec{x}, t)$ in the old frame? Prove that the conservation law is invariant under a Galilean transformation.



Galilean transform:

$$\begin{cases} \vec{r} = \vec{r}\,' + \vec{v}t, & (*) \\ t = t' \end{cases} \qquad \begin{aligned} \vec{r} &\equiv \vec{x} \\ \vec{r}\,' &\equiv \vec{x}\,' \end{aligned}$$

$$\rho'(\vec{x}\,', t) = \rho(\vec{x}, t) = \rho(\vec{x}\,' + \vec{v}t', t')$$

$$\vec{j}\,'(\vec{x}\,', t') = \vec{j}(\vec{x}, t) - \vec{v}\rho(\vec{x}, t) =$$

$$= \vec{j}(\vec{x} + \vec{v}t', t') - \vec{v}\rho(\vec{x}\,' + \vec{v}t', t')$$

Then $\frac{\partial}{\partial t'}\rho'(\vec{x}\,', t') = \frac{\partial}{\partial t'}\rho(\vec{x}\,' + \vec{v}t', t') =$

$$= \vec{v}\cdot\vec{\nabla}'\rho(\vec{x}\,' + \vec{v}t', t') + \frac{\partial}{\partial t}\rho(\vec{x}\,' + \vec{v}t', \underset{\text{acts only on this}}{t}) =$$

$$= \vec{v}\cdot\underbrace{\vec{\nabla}}_{``\vec{\nabla}'"}\rho(\vec{x}, t) + \frac{\partial}{\partial t}\rho(\vec{x}, t).$$

Using $\frac{\partial}{\partial t}\rho(\vec{x}, t) = -\vec{\nabla}\cdot\vec{j}(\vec{x}, t)$, we obtain:

$$\frac{\partial}{\partial t'}\rho'(\vec{x}\,', t') = \vec{v}\cdot\vec{\nabla}\rho(\vec{x}, t) - \vec{\nabla}\cdot\vec{j}(\vec{x}, t) =$$

$$= -\vec{\nabla}\cdot\underbrace{\left\{\vec{j}(\vec{x}, t) - \vec{v}\rho(\vec{x}, t)\right\}}_{\vec{j}\,'(\vec{x}\,', t')} = -\vec{\nabla}'\cdot\vec{j}\,'(\vec{x}\,', t').$$

So the conserv'n law is invariant under (*)

Problem 2 [10 points]

Use non-relativistic Newtonian mechanics.

a) Show that a charged particle of mass m and charge q rotates in a circle in a constant magnetic field $\vec{B}$, with the normal to the plane of the circle parallel to $\vec{B}$ (there is no electric field). Compute the angular frequency ω_C of this rotation.

EoM: $$m\frac{d\vec{v}}{dt} = \frac{q}{c}\vec{v}\times\vec{B} \quad (*)$$

gaussian units

Consider infinitesimal rotation $\delta\vec{\omega}$:

$$\delta\vec{v} = \delta\vec{\omega}\times\vec{v}$$

But from $(*)$

$$\delta\vec{v} = \underbrace{\left(-\frac{q}{mc}\right)\vec{B}\,\delta t}_{\delta\vec{\omega}} \times \vec{v}, \quad \text{or}$$

Then $$\vec{\omega}_c = -\frac{q}{mc}\vec{B}$$

cyclotron freq.

The particle velocity rotates in the plane $\perp \vec{B}$; there is no acceleration along $\vec{B}$. The position of the particle also rotates with ω_c.

b) A betatron is a device for accelerating electrons. In a betatron, electrons go about a circle of fixed radius R in a magnetic field $B(t)$ which is uniform throughout the circle but whose strength is time-dependent (the direction is perpendicular to the plane of the circle). Imagine that electrons at rest are introduced into the betatron with $B(t=0)=0$. The magnetic field is then gradually increased, and the electrons are accelerated. Prove that at time t the total magnetic flux $\Phi(t)$ within the circular orbit (all electrons follow the same orbit) is related to the value of the magnetic field $B(t)$ by

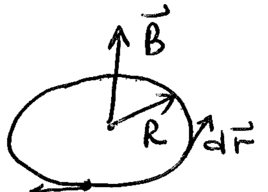
$$\Phi(t) = 2\pi R^2 B(t).$$

Consider $\bar{e}$'s kinetic energy: $\bar{e}$ charge

$$T = \frac{m v^2}{2} = \frac{m \omega_c^2 R^2}{2} = \frac{mR^2}{2}\left(\frac{e}{mc}\right)^2 \underbrace{B^2}_{\text{magnetic field}} \quad (*)$$

Here, $B = B(t) \Rightarrow$ there is induction of the electric field.

Recall that $\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$ ← gaussian units



$\vec{E}$ ← tangential

Integrating over a surface with $\bar{e}$'s orbit as its boundary, we obtain:

$$\int d\vec{s}\cdot(\vec{\nabla}\times\vec{E}) = \int_{r=R} d\vec{r}\cdot\vec{E}(\vec{r},t) =$$

$$= -2\pi R\, \underset{\substack{\uparrow \\ \text{symmetry}}}{E(R,t)} = -\frac{1}{c}\frac{d}{dt}\underbrace{\int d\vec{s}\cdot\vec{B}}_{\varphi(t)} =$$

$$= -\frac{1}{c}\frac{d\varphi(t)}{dt}.$$

Then $\underset{\text{electric field}}{E(R,t)} = \frac{1}{2\pi R c}\frac{d\varphi}{dt}.$

But $\frac{dT(t)}{dt} = e|\vec{v}||\vec{E}| = \frac{e|\vec{v}|}{2\pi R c}\frac{d\varphi}{dt} = \frac{e^2}{2\pi m c^2} B \frac{d\varphi}{dt}$

$$\frac{|\vec{v}|}{R} = |\vec{\omega}_c| = \frac{e}{mc}B(t)$$

$$(*): \frac{dT(t)}{dt} = \frac{e^2R^2}{mc^2} B \frac{dB}{dt}$$

Finally, $\frac{d\varphi}{dt} = 2\pi R^2 \frac{dB}{dt} \Rightarrow \varphi(t) = 2\pi R^2 B(t)$

$\varphi(0) = 0,\ B(0) = 0$

Problem 3 [10 points]

Find, as a function of $\vec{x}$ and t, the magnetic field in vacuum due to a point charge q with worldline $\vec{r}(t) = (vt, 0, 0)$. Show your work!

The most straightforward way is to employ Lorentz transform: $K' \to K$, where the charge is at rest in K' (at the origin):

$$\begin{cases} \vec{E}' = q \dfrac{\vec{r}'}{r'^3}, & \text{where } \vec{r}' = (x', y', z') \\ \vec{B}' = 0 & r' = \sqrt{x'^2 + y'^2 + z'^2} \end{cases}$$

Thus

$$\overset{\text{in } K'}{F'^{\alpha\beta}} = \begin{pmatrix} 0 & -E_x' & -E_y' & -E_z' \\ E_x' & 0 & 0 & 0 \\ E_y' & 0 & 0 & 0 \\ E_z' & 0 & 0 & 0 \end{pmatrix}$$

In going from K' to K, the magnetic field transforms acc. to:
(Jackson (11.148) with $\beta \to -\beta$ & primes interchanged)

$$\begin{cases} B_x = B_x' = 0, \\ B_y = \gamma(B_y' - \beta E_z') = -\beta\gamma E_z' = -\dfrac{q\gamma v z}{c(\gamma^2(x-vt)^2 + y^2 + z^2)^{3/2}} \\ B_z = \gamma(B_z' + \beta E_y') = \beta\gamma E_y' = \dfrac{q\gamma v y}{c(\gamma^2(x-vt)^2 + y^2 + z^2)^{3/2}} \end{cases}$$

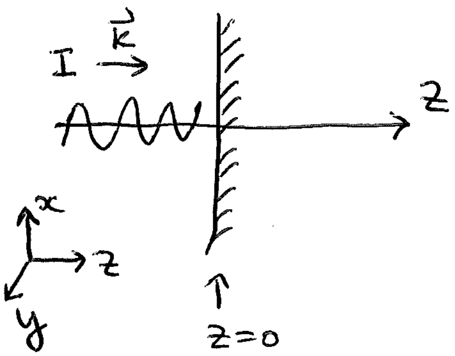
Lorentz transform:

$$\begin{cases} x' = \gamma(x - \beta c t) = \gamma(x - vt), \\ y' = y, \\ z' = z \end{cases}$$

Problem 4 [10 points]

Suppose that a plane, linearly polarized, monochromatic electromagnetic wave with frequency ω and electric field amplitude E_0 is normally incident in vacuum on a semi-infinite, *perfectly conducting* plane.

a) Write down expressions for electric and magnetic fields throughout space.



Incident wave:

$$\begin{cases} \vec{E}_I = E_0 \hat{x}\, \mathrm{Re}[e^{ikz-i\omega t}], \\ \vec{B}_I = E_0 \hat{y}\, \mathrm{Re}[e^{ikz-i\omega t}] \end{cases}$$

$$\frac{1}{c}\frac{\partial \vec{B}_I}{\partial t} + \vec{\nabla}\times\vec{E}_I = 0 \quad [\text{gaussian units}]$$

The $\vec{E}$-field must vanish at $z=0$ since the plane is perfectly conducting:

$$\vec{E}_I(z=0) = E_0 \hat{x}\, \mathrm{Re}[e^{-i\omega t}] = -\underset{\substack{\uparrow \\ \text{reflected}}}{\vec{E}_R}(z=0)$$

Thus $\vec{E}_R = -E_0 \hat{x}\, \mathrm{Re}[e^{-ikz-i\omega t}]$; $\vec{B}_R = E_0 \hat{y}\, \mathrm{Re}[e^{-ikz-i\omega t}]$

$$\vec{E} = \vec{E}_I + \vec{E}_R = E_0 \hat{x}\, \mathrm{Re}[e^{ikz-i\omega t} - e^{-ikz-i\omega t}] =$$

$$= E_0 \hat{x}\, \mathrm{Re}[2i\sin(kz)e^{-i\omega t}] =$$

$$= 2E_0 \hat{x} \sin(kz)\sin(\omega t).$$

Likewise, $\vec{B} = \vec{B}_I + \vec{B}_R = E_0 \hat{y}\, \mathrm{Re}[e^{ikz-i\omega t} + e^{-ikz-i\omega t}] =$

$$= 2E_0 \hat{y} \cos(kz)\cos(\omega t).$$

There are no fields @ $z>0$ $\Rightarrow$ the entire incoming plane is reflected.

b) Compute the time-averaged energy density in the wave to the left of the conducting plane.

Energy density $W = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) =$ ← gaussian units

$$= \frac{E_0^2}{2\pi}\left[\sin^2(kz)\sin^2(\omega t) + \cos^2(kz)\cos^2(\omega t)\right].$$

$$\langle W \rangle = \frac{E_0^2}{2\pi}\left[\sin^2(kz)\frac{1}{2} + \cos^2(kz)\frac{1}{2}\right] =$$

↑ ave. over time

$$= \frac{E_0^2}{4\pi}.$$

c) Compute the time-averaged pressure that the wave exerts on the conducting plane.

Pressure is the change in momentum per unit area per unit time.

Recall that $\vec{g}_I = \frac{1}{4\pi c}(\vec{E}_I \times \vec{B}_I)$ is the incident momentum density (gaussian units).

Since the entire wave is reflected, the momentum transfer per unit time (aka pressure P) is given by

$$P = 2c|\vec{g}_I|.$$

Thus $\langle P \rangle = \left\langle \frac{1}{2\pi}|\vec{E}_I \times \vec{B}_I| \right\rangle = \frac{E_0^2}{2\pi}\langle \cos^2(kz-\omega t)\rangle =$

↑ ave. over time

$$= \frac{E_0^2}{2\pi}\left[\cos^2(kz)\frac{1}{2} + \sin^2(kz)\frac{1}{2}\right] =$$

$$= \frac{E_0^2}{4\pi}.$$