Physics 406, Spring 2012

Final Exam
May 4, 2012

Name solutions

The ten problems are worth 10 points each.

Problem	Score
1	
2	
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1. a) Use the Heisenberg uncertainty principle and a free electron model to estimate the coherence length ξ of a superconducting electron. What is the dependence of this coherence length on the critical temperature T_C?

Use $\Delta x \Delta p \simeq \hbar$ to estimate:

$$\xi_0 \simeq \frac{\hbar}{\Delta p}$$

For a free $\bar{e}$, $E = \frac{p^2}{2m} \Rightarrow \Delta E = \frac{p \Delta p}{m}$.

We expect $\Delta E \simeq k_B T_c$ since superconducting $\bar{e}$'s are within $\Delta \simeq k_B T_c$ from E_F.

Then $\Delta p \simeq \frac{k_B T_c}{v_F} \Rightarrow \xi_0 \simeq \frac{\hbar v_F}{k_B T_c} \sim \frac{1}{T_c}$.

Hence $\xi_0 \uparrow$ as $T_c \downarrow$

b) Please write down the London equation and show how it is consistent with the perfect diamagnetism observed in superconductors.

London eq'n: $\vec{B} = -\frac{m}{n_s e^2} \vec{\nabla} \times \vec{j}_s$

Using $\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}_s$, we obtain:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = -\nabla^2 \vec{B} = \mu_0 (\vec{\nabla} \times \vec{j}_s) = -\frac{\mu_0 n_s e^2}{m} \vec{B},$$

or

$$\nabla^2 \vec{B} = \frac{\mu_0 n_s e^2}{m} \vec{B}$$

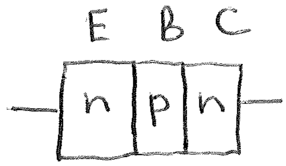
If $\vec{B} = B \hat{y}$ & B depends only on x,

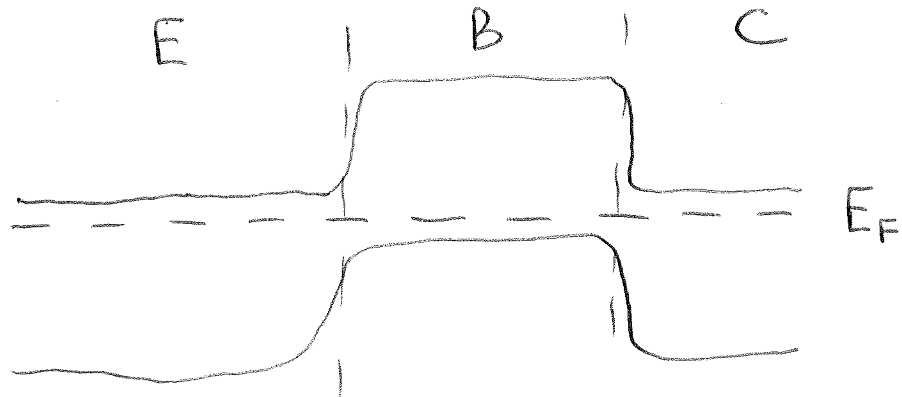
$B(x) = B_0 e^{-x/\lambda_L}$, where

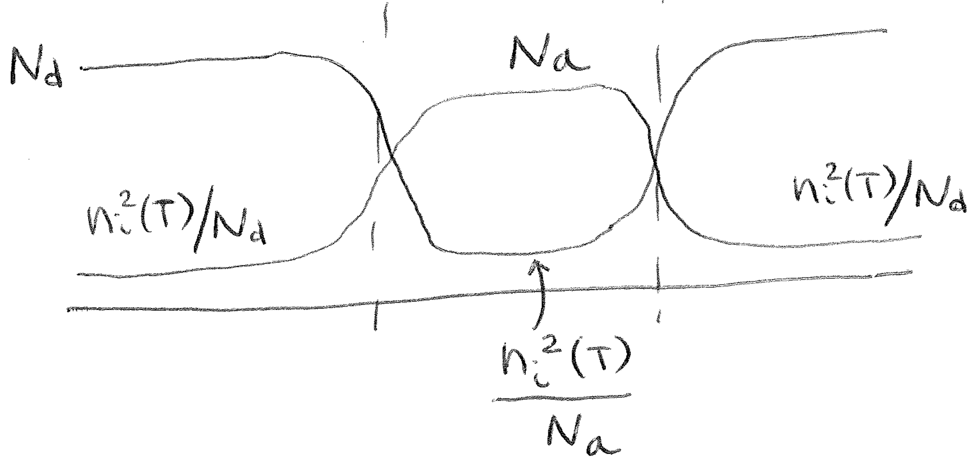
$\lambda_L = \left(\frac{m}{\mu_0 n_s e^2} \right)^{1/2}$ is the London penetration depth

Thus $B = 0$ in bulk sc $\Rightarrow$ perfect diamagnetism

2. Please sketch an energy band diagram for a n-p-n junction transistor which is not connected to any external circuits. Assuming that the concentration of impurities is N_a in the p-region and N_d in both n-regions (assume that both junctions are abrupt and that impurities are completely ionized) and that the intrinsic impurity concentration is $n_i(T)$, please sketch minority and majority carrier concentration profiles for the n-p-n transistor.







3. a) Please write down the steady-state equations for electron and hole fluxes at a p-n junction which is not connected to any external circuit. Explain the physical origin of each flux contributing to the steady-state balance.

At steady-state,

$J_{nr}^{0} = J_{ng}^{0}$ for $\bar{e}$'s, where

J_{nr}^{0} is the recombination flux ($\bar{e}$'s flow from n-region to p-region & recombine with holes there), and

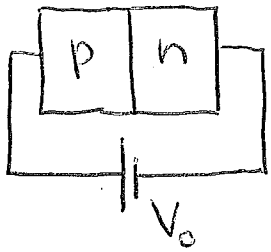
J_{ng}^{0} is the generation flux ($\bar{e}$'s are created on the p-side by thermal fluctuations & swept to the n-side by the contact potential).

Similarly,

$J_{pr}^{0} = J_{pg}^{0}$ for holes.

At steady state recombination and generation fluxes balance each other, separately for $\bar{e}$'s & holes.

b) Now the p-n junction is connected to the battery with a forward bias V_0. Please sketch the circuit diagram (including the battery and the junction), and derive an equation which expresses the total electric current I (carried by both holes and electrons) as a function of the bias voltage V_0 and the zero-voltage fluxes from part (a). Please define each flux in the equation carefully.



Forward bias

With forward bias,

$$J_{ng} = J_{ng}^0 \quad \text{but} \quad J_{nr} = J_{nr}^0 \, e^{eV_0/k_BT},$$

all generated $\bar{e}$'s are swept by the junction field anyway

since the barrier is smaller.

The $\bar{e}$ current is $I_n = e(J_{nr} - J_{ng}) =$

$$= e(\underbrace{J_{nr}^0}_{\text{“} J_{ng}^0 \text{ based on (a)}} e^{eV_0/k_BT} - J_{ng}^0) = e J_{ng}^0 (e^{eV_0/k_BT} - 1).$$

Likewise, the hole current is

$$I_p = e(J_{pr} - J_{pg}) = e(\underbrace{J_{pr}^0}_{\text{“} J_{pg}^0 \text{ based on (a)}} e^{eV_0/k_BT} - J_{pg}^0) =$$

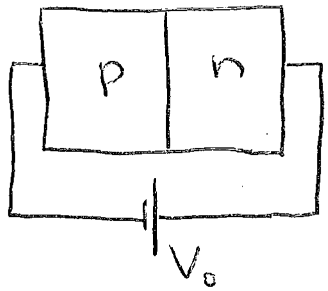
$$= e J_{pg}^0 (e^{eV_0/k_BT} - 1).$$

The total current is $I = I_n + I_p =$

$$= \underbrace{e(J_{ng}^0 + J_{pg}^0)}_{\text{“} I_0} (e^{eV_0/k_BT} - 1).$$

Note that $I \simeq I_0 e^{eV_0/k_BT}$ if $eV_0 \gg k_BT$

c) Please repeat the procedure in (b) for a reverse bias V_0.



reverse bias

Similarly to (b),

$$I = I_n + I_p = e(J_{ng} - J_{nr}) + e(J_{pg} - J_{pr}) =$$

$$J_{ng} = J_{ng}^{\circ} \qquad J_{pg} = J_{pg}^{\circ}$$

$$J_{nr} = J_{nr}^{\circ}\, e^{-eV_0/k_BT}$$

$$J_{pr} = J_{pr}^{\circ}\, e^{-eV_0/k_BT}$$

$$= e\left(J_{ng}^{\circ} - J_{nr}^{\circ}\, e^{-eV_0/k_BT}\right) +$$

$$+ e\left(J_{pg}^{\circ} - J_{pr}^{\circ}\, e^{-eV_0/k_BT}\right) =$$

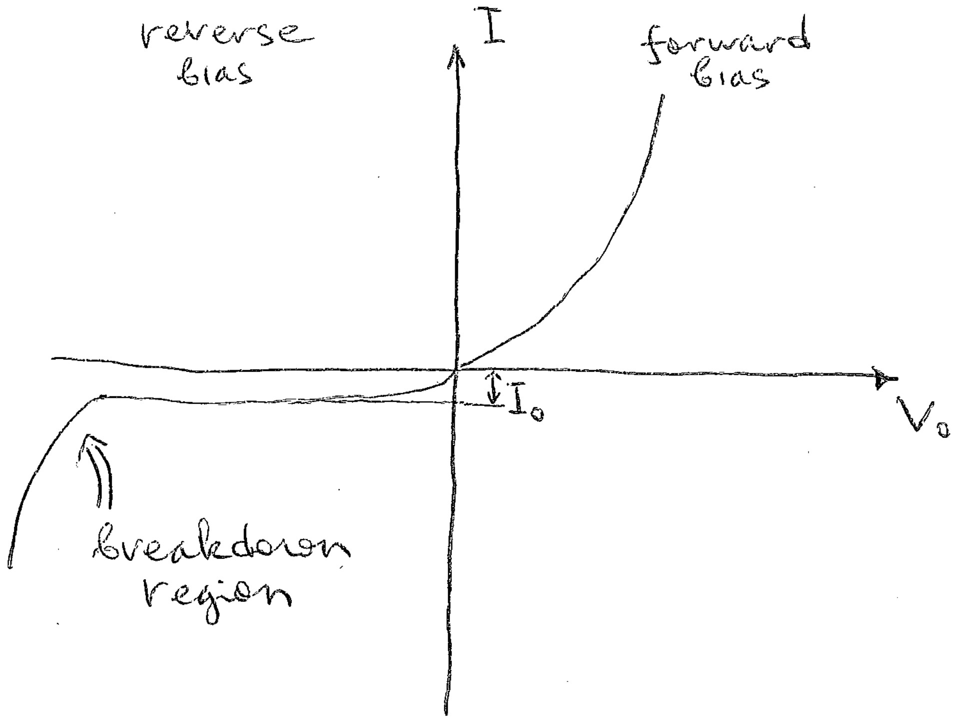
$$= \underbrace{e\left(J_{ng}^{\circ} + J_{pg}^{\circ}\right)}_{I_0}\left(1 - e^{-eV_0/k_BT}\right) =$$

$$= I_0\left(1 - e^{-eV_0/k_BT}\right).$$

Note that $I \simeq I_0$ for $eV_0 \gg k_BT \Rightarrow$

$\Rightarrow I_0$ is the saturation current.

d) Please sketch the resulting current-voltage characteristic for the p-n junction, including both forward and reverse bias regions.



4. a) Assume that a 2D metal has a simple cubic lattice with a lattice constant a = 4 Å, and that each atom has one valence electron which becomes a conduction electron in the solid. What is the concentration of conduction electrons per m^2?

In a sc lattice, there is one atom (& thus one conduction $\bar{e}$) per unit cell. Therefore

$$n = \frac{1\bar{e}}{(4\text{Å})^2} = 6.25 \times 10^{-2}\ \text{Å}^{-2} =$$

$$= 6.25 \times 10^{18}\ m^{-2}$$

b) Using the free electron model, derive the formula expressing the Fermi energy E_F as a function of electron concentration n. What is the Fermi energy for the 2D metal from part (a) (in eV)? Recall that electron mass is $m=9.1 \times 10^{-31}$ kg and Planck's constant is $\hbar=1.05 \times 10^{-34}$ J·s.

Use $\underbrace{N(k)}_{\#\text{ states up to } k} = \dfrac{\pi k^2}{\left(\frac{2\pi}{L}\right)^2}$, where L is the sample length.

Then $N = \underbrace{2}_{\text{spin}} N(k_F) = 2\dfrac{k_F^2 A}{4\pi}$, where $A = L^2$.

Thus $k_F^2 = 4\dfrac{\pi}{2}\underbrace{n}_{\frac{N}{A}\text{ - } \bar{e} \text{ concentr'n}}$

$$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m} \times (2\pi n) = \frac{\hbar^2 \pi n}{m}$$

From part (a),

$$E_F = \frac{(1.05 \times 10^{-34}\ \mathrm{J \cdot s})^2}{9.1 \times 10^{-31}\ \mathrm{kg}}\ \pi \times (6.25 \times 10^{18}\ \mathrm{m^{-2}}) \simeq$$

$$\simeq 2.38 \times 10^{-19}\ \mathrm{J} \simeq 1.49\ \mathrm{eV}.$$

5. a) The average energy of a quantum-mechanical oscillator at equilibrium with temperature T is given by

$\langle \varepsilon \rangle = \hbar\omega/(\exp(\hbar\omega/ k_BT) - 1)$,

where $\hbar\omega$ is the energy difference between adjacent energy levels of an isolated oscillator. In the framework of the Einstein theory of specific heat, write down the internal energy per unit volume of the solid. The concentration of atoms in a solid is n.

In Einstein theory, there is only one ω, ω_E. Then the internal energy per unit volume is simply

$$E = \underbrace{3}_{\text{\# DoF per atom}}\,\underbrace{n}_{\text{atomic concentr'n}}\,\frac{\hbar\omega_E}{e^{\hbar\omega_E/k_BT} - 1}$$

b) Derive the formula for the specific heat C_V in the Einstein model. What is the behavior of the specific heat as $T \rightarrow 0$ (such that $k_B T \ll \hbar\omega$)? How does this behavior compare with experimental findings? What is the limiting value of the specific heat as $T \rightarrow +\infty$ (such that $k_B T \gg \hbar\omega$), and why?

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V,$$ so that

$$C_V = 3n\hbar\omega_E\left[-\frac{1}{\left(e^{\hbar\omega_E/k_BT}-1\right)^2}\right]\frac{\hbar\omega_E}{k_B}\times$$

$$\times\left(-\frac{1}{T^2}\right)e^{\hbar\omega_E/k_BT} =$$

$$= 3nk_B\left(\frac{\hbar\omega_E}{k_BT}\right)^2\frac{e^{\hbar\omega_E/k_BT}}{\left(e^{\hbar\omega_E/k_BT}-1\right)^2}.$$

This is C_V per unit volume $\Rightarrow$
$\Rightarrow$ replace n with N_A (avogadro #) to get C_V per mole.

Then $$C_V = 3\underbrace{R}_{N_Ak_B}\left(\frac{\hbar\omega_E}{k_BT}\right)^2\frac{e^{\hbar\omega_E/k_BT}}{\left(e^{\hbar\omega_E/k_BT}-1\right)^2}.$$

If $k_BT \ll \hbar\omega_E$,

$$C_V \simeq 3R\left(\frac{\hbar\omega_E}{k_BT}\right)^2 e^{-\hbar\omega_E/k_BT}.$$

C_V approaches $T=0$ exponentially, much faster than experimentally observed.

If $k_BT \gg \hbar\omega_E$, $C_V \simeq 3R$ – the classical limit expected acc. to the equipartition theorem.

6. a) Imagine a 1D random walker which starts from $x_0=0$ and takes a step of fixed length L to the right or to the left with equal probabilities (the probability to stay in the same position is 0). Derive the average displacement, $<x_N>$, after N steps for an ensemble of such walkers. Now repeat this calculation for an ensemble of random walkers in which the probability to go to the left, p_L, is less than the probability to go to the right, p_R. Express your answer in terms of p_R-p_L.

Introduce a random variable

$$k = \begin{cases} +1, & p_R \\ -1, & p_L \end{cases} \qquad p_R + p_L = 1$$

Then $x_N = x_{N-1} + kL$. ← step length

The average displacement is

$$\langle x_N \rangle = \langle x_{N-1} \rangle + L \underbrace{\langle k \rangle}_{\substack{\| \\ (+1)\times p_R + (-1)\times p_L = \\ = p_R - p_L}}$$

If $p_R = p_L = 1/2$ (unbiased random walk),

$$\langle x_N \rangle = \langle x_{N-1} \rangle \Rightarrow \underline{\underline{\langle x_N \rangle = x_0 = 0}}.$$

$\langle k \rangle = 0$

the CM of the ensemble of particles does not move

In general, $\langle x_N \rangle = \langle x_{N-1} \rangle + L(p_R - p_L) \Rightarrow$

$$\Rightarrow \underline{\underline{\langle x_N \rangle = NL(p_R - p_L)}}.$$

b) Derive the average squared displacement, $<x_N^2>$, after N steps for the ensemble of unbiased random walkers from part (a) ($p_L = p_R$). What is the dependence of $<x_N^2>$ on the number of steps N?

Using $x_N = x_{N-1} + kL$ as in part (a),

$$\langle x_N^2 \rangle = \langle (x_{N-1} + kL)^2 \rangle = \langle x_{N-1}^2 \rangle +$$

$$+ 2L \underbrace{\langle x_{N-1} k \rangle}_{\langle x_{N-1} \rangle \langle k \rangle = 0} + L^2 \underbrace{\langle k^2 \rangle}_{=1} =$$

After N-1 steps, equal prob. to go left or right, so will cancel out

k^2 same for steps to the left & to the right

$$= \langle x_{N-1}^2 \rangle + L^2 .$$

So, $\langle x_N^2 \rangle = \langle x_{N-1}^2 \rangle + L^2 =$

$$= \langle x_{N-2}^2 \rangle + 2L^2 = \ldots =$$

$$= \underbrace{\langle x_0^2 \rangle}_{= x_0^2 = 0} + NL^2 .$$

$x_0^2 = 0$, all walkers start at origin

$\langle x_N^2 \rangle = NL^2 \underline{\underline{\sim N}}$, grows linearly with N.

7. a) What is the average energy of an electron in a 3D Fermi gas at T=0 K? Please express your answer in terms of the Fermi energy E_F. Hint: you may want to average the electron energy over a sphere in k-space.

$$E(k) = \frac{\hbar^2 k^2}{2m}$$

$$\langle E(k) \rangle = \frac{\hbar^2}{2m} \frac{\int_0^{k_F} dk\, k^4}{\int_0^{k_F} dk\, k^2} =$$

$$= \frac{\hbar^2}{2m} \frac{3}{5} k_F^2 = \frac{3}{5} E_F .$$

b) Using results from part (a), find the average electron energy in silver at T=0 K (note that E_F = 5.51 eV in silver).

In silver,

$$\langle E(k) \rangle = \frac{3}{5} \times 5.51\,eV \simeq 3.31\,eV .$$

c) What is the temperature of an ideal gas with the same value of the average molecular energy? (Note that k_B=1.38 x 10^{-23} J/K = 8.6 x 10^{-5} eV/K)

$\langle E \rangle \sim k_B T$ in an ideal gas

$\Downarrow$

$$T \sim \frac{3.31\ eV}{8.6 \times 10^{-5}\ eV/K} \simeq 3.8 \times 10^4\ K$$

8. Consider a 2D array of atoms arranged in a regular square lattice, with a lattice constant of 3.6 Å.

a) Assuming that the phonon dispersion is well described by the Debye approximation ($\omega = ck$), find the phonon density of states $g(\omega)$ for this 2D solid.

In 2D,

$$N(k) = \frac{\pi k^2}{\left(\frac{2\pi}{L}\right)^2} = \frac{Ak^2}{4\pi},$$

where $A = L^2$ is the area of the solid.

Since $\omega = ck$, $N(\omega) = \dfrac{A\omega^2}{4\pi c^2}$.

Finally,

$$\underbrace{D(\omega)}_{g(\omega),\ \text{DoS}} = \overset{\#\text{ modes in a 2D solid}}{2}\frac{dN}{d\omega} = \frac{A\omega}{\pi c^2}$$

b) If the speed of sound is c = 10^3 m/s, what is the Debye temperature of this 2D solid? Hint: use the density of states from part (a). You may need these constants: $\hbar$=1.05x10^{-34} J·s, k_B=1.38x10^{-23} J/K.

The cutoff frequency is defined as

$$\int_0^{\omega_D} d\omega\, g(\omega) = 2N \quad \text{in 2D, or}$$

$$\frac{A\omega_D^2}{4\pi c^2} = N \Rightarrow \underset{\substack{\uparrow \\ \text{Debye freq.}}}{\omega_D} = \left[4\pi c^2\left(\frac{N}{A}\right)\right]^{1/2}.$$

atoms per unit area

$$\frac{N}{A} = \frac{1}{(3.6\,\text{Å})^2} \simeq 7.7\times10^{-2}\ \text{Å}^{-2} = 7.7\times10^{18}\ m^{-2}.$$

Thus $\omega_D = 2\times10^3 \frac{m}{s}\sqrt{\pi}\sqrt{7.7\times10^{18}\ m^{-2}} \simeq$

$$\simeq 9.84\times10^{12}\ s^{-1}.$$

The Debye T is defined by

$k_B\theta_D = \hbar\omega_D$, so that

$$\theta_D = \frac{1.05\times10^{-34}\ J\cdot s}{1.38\times10^{-23}\ J/K}\times\left(9.84\times10^{12}\ \frac{1}{s}\right) \simeq 74.9\ K.$$

9. a) An incident monochromatic X-ray beam with wavelength λ = 1.7 Å is reflected from the (111) plane in a 3D solid with a Bragg angle of 26° for the n=1 reflection. Please compute the distance (in Å) between adjacent (111) planes.

Bragg's law:

$$\underbrace{n}_{\text{1 here}}\lambda = 2d\sin\theta$$

In our case,

$$d_{111} = \frac{\lambda}{2\sin\theta} = \frac{1.7\text{ Å}}{2\times\sin(26°)} \simeq \underline{\underline{1.94\text{ Å}}}.$$

b) Assuming that the solid has an fcc lattice, use the result from part (a) to compute the lattice constant (in Å).

In the fcc lattice,

$$d_{111} = \frac{a}{\sqrt{1^2+1^2+1^2}}, \text{ or}$$

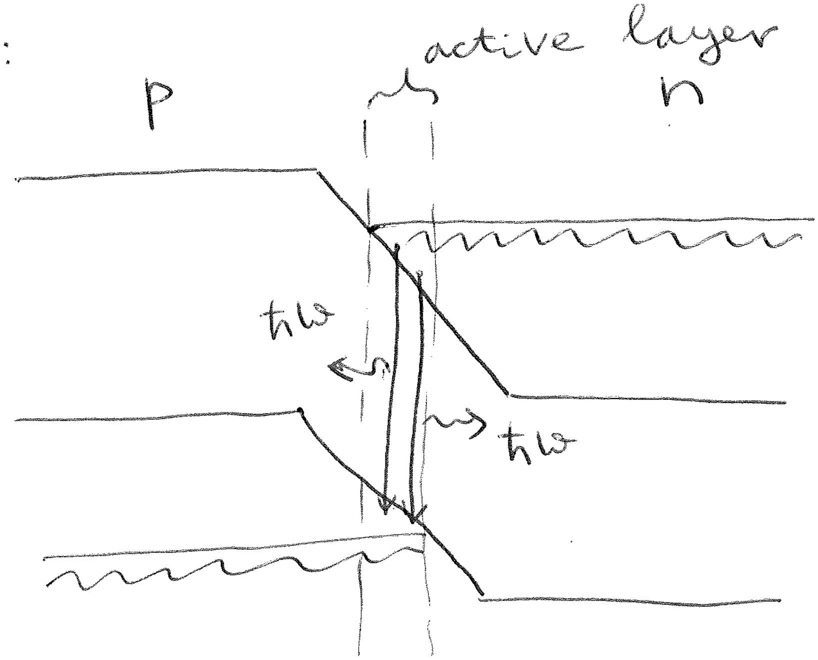
$$a = d_{111}\sqrt{3} = 1.94\text{ Å}\times\sqrt{3} \simeq \underline{\underline{3.36\text{ Å}}}.$$

Recall that

$$d_{hkl} = \frac{a}{\sqrt{h^2+k^2+l^2}}$$

10. Please describe the basic idea behind a semiconductor laser. How is it different from an LED light, and what are the similarities? What is the purpose of bandgap engineering in semiconductor lasers? Please use a sketch of the laser energy band diagram to justify your answers.

In semiconductor lasers, a heavily doped p-n junction is used to create population inversion in the active layer, as demonstrated in the sketch below:



Note that photon emission is due to interband transitions and that the system is out of equilibrium (ē's are continuously injected on the right, recombining w/holes in the active layer). The basic design of an LED is the same, but the LED operates below the lasing threshold, and no coherent amplification occurs. Finally, bandgap engineering is used to modulate the frequency of the emitted light, both in LEDs & lasers.