

Physics 406, Spring 2012

Midterm II

April 9, 2019

Name solutions

The five problems are worth 20 points each.

Problem	Score
1	
2	
3	
4	
5	
Total	

1. Consider a two-dimensional (2D) film of atoms arranged in a regular simple cubic lattice. The lattice spacing is 2.5 Å. Let us assume that each atom contributes two "valence" electrons that act as free electrons in the 2D solid.

a) What is the Fermi energy (in eV) of this 2D solid at T=0? Recall that

$$\frac{\hbar^2}{2m_e} = 3.845 \text{ eV} \cdot \text{Å}^2$$

The total # of  $\bar{e}$ 's

$$N = \int_0^{E_F} dE D(E), \text{ where}$$

$$D(E) = \underset{\substack{\uparrow \\ \text{spin}}}{2} \frac{dN}{dE} = 2 \frac{dN}{dk} \frac{dk}{dE} \text{ is the DoS.}$$

$$\text{In 2D, } N(k) = \frac{\pi k^2}{\left(\frac{2\pi}{L}\right)^2} = \frac{L^2 k^2}{4\pi}$$

↓ free  $\bar{e}$ 's

$$E = \frac{\hbar^2 k^2}{2m} \Rightarrow \frac{dE}{dk} = \frac{\hbar^2 k}{m}$$

↑  $\bar{e}$  mass

$$\text{Then } D(E) = 2 \frac{L^2 k}{2\pi} \frac{m}{\hbar^2 k} = \frac{L^2 m}{\pi \hbar^2} = \text{const}(E)$$

$$\text{Thus } N = \frac{L^2 m}{\pi \hbar^2} E_F, \text{ or}$$

$$E_F = \frac{N}{L^2} \frac{\pi \hbar^2}{m}$$

↙ n -  $\bar{e}$  concentr'n

$$n = \frac{2\bar{e}}{(2.5 \text{ Å})^2} = 0.32 \text{ Å}^{-2}$$

$$\text{Finally, } E_F = n \frac{\pi \hbar^2}{m} = 0.32 \text{ Å}^{-2} \times (2\pi)^2 \times 3.845 \text{ eV} \cdot \text{Å}^2 = 7.73 \text{ eV.}$$

b) Please calculate the average energy per electron in this 2D film at  $T=0$ .

Per unit area,

$$\frac{1}{L^2} \langle E \rangle = \frac{1}{L^2} \int_0^{E_F} dE E D(E) = \frac{E_F^2}{2} \underbrace{\frac{L^2 m}{\pi \hbar^2}}_{\frac{N}{E_F}} \frac{1}{L^2} =$$

$$= \frac{E_F}{2} n$$

Per  $\bar{e}$ ,

$$\frac{\langle E \rangle}{n} = \frac{E_F}{2}, \text{ as expected since } \underline{\underline{D(E) = \text{const}(E)}}.$$

c) At room temperature ( $T=300\text{K}$ ,  $k_B T=0.025\text{ eV}$ ), what is the probability that states  $0.05\text{ eV}$  and  $0.2\text{ eV}$  above the Fermi level are occupied by an electron? Please use both full Fermi-Dirac function and its Boltzmann-like expansion, and compare the two results. Which energy level yields the best agreement between the full and the approximate calculation, and why?

For  $0.05\text{ eV}$ ,

$$f_{\text{FD}}(E) = \frac{1}{e^{(E-E_F)/k_B T} + 1} = \frac{1}{e^2 + 1} \approx 0.12$$

$$f^{\text{B}}(E) \approx e^{-(E-E_F)/k_B T} = e^{-2} \approx 0.14$$

For  $0.2\text{ eV}$ ,

$$f^{\text{FD}}(E) = \frac{1}{e^8 + 1} \approx 3.4 \times 10^{-4}$$

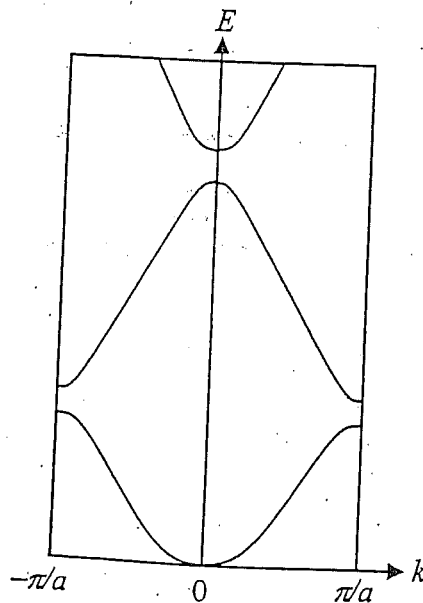
$$f^{\text{B}}(E) \approx e^{-8} \approx 3.4 \times 10^{-4}$$

$0.2\text{ eV}$  yields better agreement since  $\frac{E-E_F}{k_B T} = 8 \gg 1$ , making the Boltzmann expansion accurate. 3

2. The diagram below shows the electronic band structure of some hypothetical one-dimensional (1D) solid. Will this solid be a metal or an insulator at  $T=0$  if it has:

- a) 1 electron/primitive unit cell?
- b) 2 electrons/primitive unit cell?
- c) 3 electrons/primitive unit cell?
- d) 4 electrons/primitive unit cell?

Please justify your answers!



metal

$1 \bar{e}/\text{cell} \Rightarrow N \bar{e}'\text{'s total} \Rightarrow$  1st band half-full  
 $\uparrow$   
 # cells

$3 \bar{e}/\text{cell} \Rightarrow 3N \bar{e}'\text{'s} \Rightarrow$  1st band full, 2nd band half-full

} metal

$2 \bar{e}/\text{cell} \Rightarrow 2N \bar{e}'\text{s} \Rightarrow \text{1st band full}$   
 $4 \bar{e}/\text{cell} \Rightarrow 4N \bar{e}'\text{s} \Rightarrow \text{1st, 2nd bands full}$   
} insulator

Key points:

- 1) Each band accommodates  $2N \bar{e}'\text{s}$ 

$\downarrow$  spin
- 2)  $x \bar{e}'\text{s}$  per primitive unit cell  $\Rightarrow$   
 $\Rightarrow Nx$  total  $\bar{e}'\text{s}$
- 3) Full bands cannot carry electric current. Metals have to have partially occupied bands.

3. Please describe recent developments in the field of ultra-cold Fermi gases. Include some current challenges and future goals in your description, and discuss the differences between Fermi and Bose gases at ultra-low temperatures. Please use a minimum of 4 sentences in your response.

at ultra-low  $T$ s, bosons condense into the ground state, forming Bose-Einstein condensate. This is not possible for fermions due to Pauli exclusion principle, which makes it impossible to achieve ultra-low  $T$ s through head-on "s-wave" collisions in Fermi gases. However, cooling is achieved if mixtures of atoms in different states undergo s-wave collisions. The mixture can ~~be~~ contain atoms in two distinct spin states, or isotopes. Fermionic condensates could exhibit superconductivity and superfluidity, as well as other interesting quantum properties.

4. Consider a 1D biased random walk in which a step of variable length is made either to the left or to the right. The time between successive steps is fixed at  $\Delta t$ . Each step has a length of  $kL$ , where  $k$  is a random variable:

$$k = \pm 1, \pm 2, \dots$$

$P(k)$  is the probability of  $k$  (and thus the probability of a step length  $kL$ ), and

$$u = \langle k \rangle = \sum_{k=\pm 1, \pm 2, \dots} k P(k) \neq 0$$

is the drift velocity. Recall that variance of a random variable  $x$  is given by

$$\text{Var}(x) = \langle x^2 \rangle - \langle x \rangle^2$$

Derive an expression for the variance of the displacement  $x_N$  after  $N$  steps, and discuss its dependence on the total time  $t$  elapsed since the start of the walk. Define the diffusion constant  $D$  by analogy with the unbiased random walk. How does it depend on the variance of  $k$ ?

$$\begin{aligned} \langle x_N^2 \rangle &= \langle (x_{N-1} + kL)^2 \rangle = \\ &= \langle x_{N-1}^2 \rangle + 2L \langle x_{N-1} \rangle \langle k \rangle + L^2 \langle k^2 \rangle. \end{aligned}$$

$$\langle x_N \rangle = \langle x_{N-1} \rangle + L \langle k \rangle.$$

Then

$$\begin{aligned} \text{Var}(x_N) &= \langle x_N^2 \rangle - \langle x_N \rangle^2 = \\ &= \langle x_{N-1}^2 \rangle + 2L \langle x_{N-1} \rangle \langle k \rangle + L^2 \langle k^2 \rangle - \\ &\quad - \langle x_{N-1} \rangle^2 - 2L \langle x_{N-1} \rangle \langle k \rangle - L^2 \langle k^2 \rangle = \\ &= \text{Var}(x_{N-1}) + L^2 \text{Var}(k). \end{aligned}$$

Thus

$$\text{Var}(x_N) = NL^2 \text{Var}(k).$$

We recall that  $N = \frac{t}{\Delta t}$ , and

define  $D = \frac{L^2 \text{Var}(k)}{2\Delta t}$  to obtain

$$\text{Var}(x_N) = 2Dt, \text{ where}$$

$D$  is the diffusion constant.

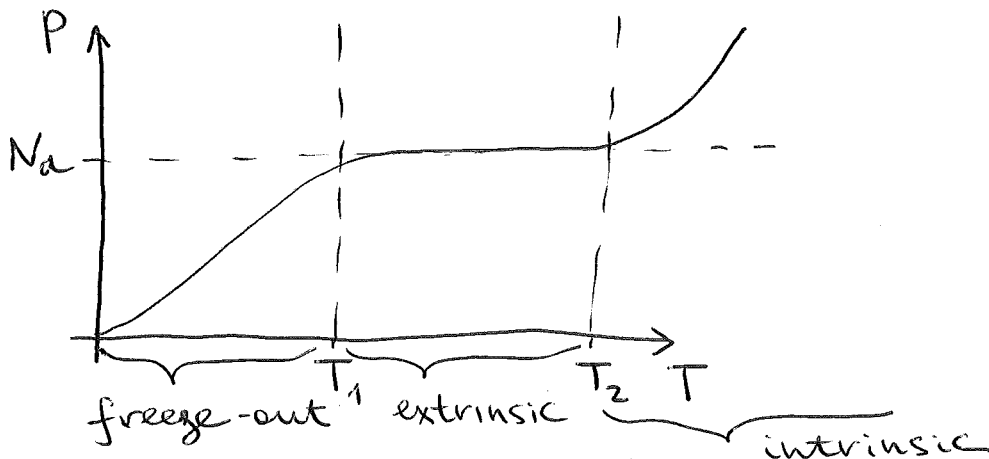
Note that  $D \sim \text{Var}(k)$ , so that diffusion is faster for steps w/ higher variance.



5. a) What is a semiconductor and how is it distinct from metals and insulators?

A semiconductor is a material with  $E_g \leq 2 \text{ eV}$ , where  $E_g$  is the gap between VB and CB. At or above room T, enough  $\bar{e}$ 's are excited into CB to make it a conductor (unlike insulators). However, the conductivity is much smaller than with metals since relatively few  $\bar{e}$ 's are in the CB.

b) Please sketch  $p(T)$  vs.  $T$  (temperature dependence of the carrier concentration) for a p-type semiconductor. Indicate on the plot and explain the three main temperature regimes.



$T < T_1$ :  $\bar{e}$ 's fall back from acceptor level to VB, leaving neutral acceptors & recombining w/holes in the VB.

$T_1 < T < T_2$ : extrinsic regime,  $p \approx N_a$ . All acceptors ionized, holes in VB.

$T > T_2$ : Thermal  $\text{VB} \rightarrow \text{CB}$  transitions dominate impurity effects  $\Rightarrow \Rightarrow$  intrinsic regime

c) Use the Maxwell-Boltzmann approximation to the Fermi-Dirac distribution to compute the electron concentration as a function of Fermi energy and gap energy in the intrinsic semiconductor regime. Use

$$g_e(E) = \frac{1}{2\pi^2} \left( \frac{2m_e}{\hbar^2} \right)^{3/2} (E - E_g)^{1/2}, \text{ where } E_g \text{ is gap energy.}$$

for the density of states on the bottom of the conduction band.

Recall that  $\int_0^{\infty} dx x^{1/2} e^{-x} = \frac{\sqrt{\pi}}{2}$ .

Using  $f(E) \approx e^{E_F/k_B T} e^{-E/k_B T}$

$E_{c2} \rightarrow +\infty$  extend upper limit since integrand is very small

$$n = \int_{E_g}^{\infty} dE f(E) g_e(E) \quad \text{⊖}$$

$\uparrow$  top of VB has  $\phi$  energy

$$\text{⊖} \quad \frac{1}{2\pi^2} \left( \frac{2m_e}{\hbar^2} \right)^{3/2} \int_{E_g}^{\infty} dE e^{E_F/k_B T} e^{-E/k_B T} (E - E_g)^{1/2} =$$

$x = \frac{E - E_g}{k_B T}$

$$= \frac{1}{2\pi^2} \left( \frac{2m_e}{\hbar^2} \right)^{3/2} e^{E_F/k_B T} e^{-E_g/k_B T} (k_B T)^{3/2} \times$$

$$\times \underbrace{\int_0^{\infty} dx x^{1/2} e^{-x}}_{\frac{\sqrt{\pi}}{2}} = 2 \left( \frac{m_e k_B T}{2\pi \hbar^2} \right)^{3/2} e^{E_F/k_B T} e^{-E_g/k_B T}$$