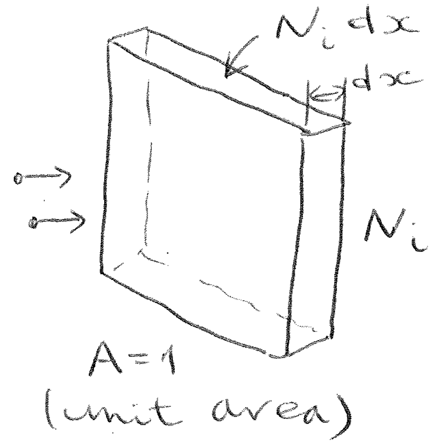
HW #4 solutions

Physics 406

1. θ Pr. 2

MFP $\ell = \tau v$, such that $\frac{dx}{\ell}$ is the probability to have a collision in going from x to $x+dx$.
mean free path



Consider a particle which travels distance dx through material with N_i impurities per unit volume. Each impurity has a cross-section (effective collision area) σ_i. Then the total area covered by the scatterers is $\sigma_i N_i dx$ $(A=1)$.

Thus
$$\underbrace{\sigma_i N_i dx}_{\text{prob. to have a collision}} = \frac{dx}{\ell} \Rightarrow \ell = \frac{1}{\sigma_i N_i}$$

2. a) $\bar{e}$ EoM:

$$m\frac{dv}{dt} = -eE - m\frac{v}{\tau} = 0 \text{ @ steady state}$$

$$\Downarrow$$

$$\underbrace{\langle v \rangle}_{v_d \text{ drift velocity}} = -\frac{eE\tau}{m}.$$

Current density

$$j = -nev_d = \underbrace{\frac{ne^2\tau}{m}}_{\sigma} E = \sigma E,$$

Ohm's law

b) We have shown that

$$K = \frac{1}{3} v_F C_v l_F, \text{ where}$$

K: thermal conduct.

v_F, l_F: @ Fermi level

$$C_v = \frac{\pi^2}{2} R \frac{T}{T_F} = \frac{\pi^2}{2} \frac{N k_B^2 T}{E_F} = \frac{\pi^2 N k_B^2 T}{m v_F^2}.$$

Then $$K = \frac{\pi^2 N k_B^2 T \tau_F}{3m}.$$

Using $$\sigma = \frac{Ne^2\tau_F}{m},$$

$$\frac{K}{\sigma T} = \frac{1}{3}\left(\frac{\pi k_B}{e}\right)^2 \equiv L, \text{ the Lorenz \#}$$

e: $\bar{e}$ charge

(3.) The Fermi-Dirac distribution

$$f(E) = \frac{1}{e^{(E-E_F)/k_BT} + 1}$$

becomes Boltzmann distr'n

$$f(E) \simeq e^{-(E-E_F)/k_BT}$$

if $e^{(E-E_F)/k_BT} \gg 1$, or

$$E - E_F \gg k_BT \quad (E > E_F).$$

Thus the Boltzmann expansion is valid in the tail of the FD distr'n.

4.

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Psi(x,y,z) =$$

$$= E\,\Psi(x,y,z).$$

BC's : (box of Side L)

$$\begin{cases} \Psi(0,y,z) = \Psi(L,y,z) = 0, \\ \Psi(x,0,z) = \Psi(x,L,z) = 0, \\ \Psi(x,y,0) = \Psi(x,y,L) = 0. \end{cases}$$

Let's try

$$\Psi = A\sin(k_x x) * \sin(k_y y) * \sin(k_z z).$$

Half of BC's satisfied automatically

We get:

$$\frac{\hbar^2}{2m}\underbrace{(k_x^2+k_y^2+k_z^2)}_{|\vec{k}|^2 = k^2}\Psi = E\Psi, \text{ or}$$

$$E = \frac{\hbar^2 k^2}{2m} \text{ as expected.}$$

The remaining BC's:

$$\begin{cases} \sin(k_x L) = 0, \\ \sin(k_y L) = 0, \\ \sin(k_z L) = 0 \end{cases} \Rightarrow \begin{cases} k_x = \frac{n_x \pi}{L}, \\ k_y = \frac{n_y \pi}{L}, \\ k_z = \frac{n_z \pi}{L} \end{cases} \quad n_{x,y,z} = 0, \pm 1, \ldots$$

So, $\Psi = A\sin\left(\frac{n_x \pi x}{L}\right)\sin\left(\frac{n_y \pi y}{L}\right)\sin\left(\frac{n_z \pi z}{L}\right)$,

where A is found from $\int dV\,|\Psi|^2 = 1$

$\int_0^L dx\,\sin^2\left(\frac{n_x \pi x}{L}\right) = \frac{L}{2}$, same for y & z:

$$\underline{\underline{A = \left(\frac{2}{L}\right)^{3/2}}}$$

5. ① chapter 4, Problem 6.

T/T_F for $T = 300$ for Cu, Na and Ag.

From Table 4.1 in Omar we have

Element	E_F (eV)
Cu	7.0
Na	3.1
Ag	5.5.

$$E_F = k_B T_F \Rightarrow T_F = E_F / k_B.$$

$$k_B = 1.4 \times 10^{-23} \text{ J/K} = 8.6 \times 10^{-5} \text{ eV/K}$$

Element	T_F	$T/T_F \big|_{T=300}$
Cu	8×10^4	3.8×10^{-3}.
Na	3.6×10^4	8.3×10^{-3}
Ag	6.4×10^4	4.6×10^{-3}.

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6. 0 Chapter 4 Problem 7.

Fraction of electrons excited above Fermi level at $T \sim 300$ K $\sim \dfrac{k_B T}{E_F}$

Cu $\Rightarrow$ $E_F = 7.0$ eV $\Rightarrow$ $f = \dfrac{(8.6 \times 10^{-5})(300)}{7}$

$\Downarrow$

$f_{Cu} = 3.7 \times 10^{-3}$

Na $\Rightarrow$ $E_F = 3.1$ eV $\qquad f_{Na} = 8.3 \times 10^{-3}$

7. Kittel chap. 6 problem 4.

a) N_e = # electrons

N_p $\downarrow$ # protons in sun $\sim \dfrac{M_\odot}{m_p}$

$$= \frac{2 \times 10^{33}\ g}{1.7 \times 10^{-24}\ g} \sim 10^{57}$$

Let us assume that there are roughly the same number of e^-'s and p's

$$\Downarrow$$

$$\varepsilon_F = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}$$

where

$$n = \frac{N_e}{V} \qquad \text{where} \quad V = \frac{4}{3} \pi R_s^3.$$

$$= \frac{4}{3} \pi (2 \times 10^9)^3$$

$$\Downarrow$$

$$\sim 3 \times 10^{28}$$

$$n \sim \frac{10^{57}}{3 \times 10^{28}} \sim 3 \times 10^{28} \text{ electrons}/\text{cm}^3.$$

$$\Downarrow$$

$$\varepsilon_F \sim \frac{(10^{-27})^2}{2(9 \times 10^{-28})} (3\pi^2 (3 \times 10^{28}))^{2/3}$$

$$\sim \frac{1}{2}(10^{-27})(10^{20}) \sim 5 \times 10^{-6} \text{ ergs}$$

$$\varepsilon_F \sim 5 \times 10^{-6} \text{ ergs} \times \frac{1 \text{ eV}}{1.6 \times 10^{-12} \text{ ergs}}$$

$$\Downarrow$$

$$\boxed{\varepsilon_F = 3.6 \times 10^4 \text{ eV}}$$

b) k_F is not affected by relativity

In 3d we determine k_F

$$\Downarrow$$

$$N = \frac{3 \cdot \frac{4}{3} \pi k_F^3}{(2\pi)^3 / V} \Rightarrow k_F \sim \left(\frac{N}{V}\right)^{1/3}$$

$$\Downarrow$$

In the relativistic limit

$$\boxed{\varepsilon_F = \hbar k_F c \sim \hbar c \left(\frac{N}{V}\right)^{1/3}}$$

c) Now $\tilde{R}_s = 10 \text{ km} = 10^6 \text{ cm}$

$(R_s = 2 \times 10^9 \text{ cm})$

$$n \sim 3 \times 10^{28} \frac{e}{\text{cm}^3} \times \frac{(2 \times 10^9)^3}{10^{18}}$$

$$\sim 2.4 \times 10^{38} \; e/\text{cm}^3.$$

$$\varepsilon_F \sim \hbar c n^{1/3} \sim (10^{-27})(3 \times 10^{10})(10^{13})$$

$$\sim 2 \times 10^{-4} \text{ erg} \times \frac{1 \text{ eV}}{1.6 \times 10^{-12} \text{ erg}}$$

$$\Downarrow$$

$$\boxed{\varepsilon_F \sim 10^8 \text{ eV}}$$ relativistic

$(m_e/c^2 \sim .51 \times 10^6 \text{ eV})$

8. Kittel chapter 6 Problem 5.

$$\varepsilon_F = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}$$

$$\rho = .081 \ g/cm^3$$

$$\frac{\#\ \text{moles}}{cm^3} = \frac{1}{3}\ 81 \times 10^{-3}$$

$$= 27 \times 10^{-3} = 2.7 \times 10^{-2} \ \frac{\text{moles}}{cm^3}.$$

n_H = concentration of atoms $= 2.7 \times 10^{-2} \ \frac{\text{moles}}{cm^3} \times 6 \times 10^{23} \frac{\text{atoms}}{\text{moles}}$

$$= 1.6 \times 10^{22} \text{ atoms}/cm^3.$$

$$m_H = (3)\, m_p = (3)\,(1.6 \times 10^{-24}\ g)$$

$$\sim 5 \times 10^{-24}\ g.$$

$$\varepsilon_F \sim \frac{(10^{-27})^2}{2.5 \times 10^{-24}} \left(3 \cdot \pi^2 \left(1.6 \times 10^{22}\right)\right)^{2/3}.$$

$$\sim \frac{10^{-54}}{10^{-23}} \left[(3\pi^2)(16)(10^{21}) \right]^{2/3}$$

$$\sim \underbrace{10^{-31}} \cdot \left[[30][16] \right]^{2/3} \cdot 10^{14}$$

$$\boxed{\varepsilon_F \sim 6 \times 10^{-16} \text{ erg}}$$

$$\boxed{T_F = \frac{\varepsilon_F}{k_B} \sim \frac{6 \times 10^{-16} \text{ erg}}{1.4 \times 10^{-16} \text{ erg}/K} \sim 4.29 \text{ K}.}$$

(9.) The article describes how to create a low-T gas of fermionic atoms. Whereas bosons fall into the ground level at low T, fermions cannot share quantum states due to the Pauli exclusion principle. Thus cooling through head-on "s-wave" collisions is not possible for fermions. To circumvent this difficulty, Jin's group used atoms in two distinct spin states. Hulet's group used mixtures of isotopes. In both cases quantum degeneracy was achieved via collisions, using evaporative cooling.