Physics 406, Spring 2011

Final Exam
May 6, 2011

Name solutions

The ten problems are worth 10 points each.

Problem	Score
1	
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1. Use the Heisenberg uncertainty principle and a free electron model to estimate the coherence length ξ of a superconducting electron. What is the dependence of this coherence length on the critical temperature T_C?

We use $\Delta x \, \Delta p \simeq \hbar$ to estimate the coherence length:

$$\xi \simeq \frac{\hbar}{\Delta p}$$

For a free $\bar{e}$, $E = \frac{p^2}{2m}$, or

$$\Delta E = \frac{p \Delta p}{m}.$$

Since superconducting $\bar{e}$'s lie within $k_B T_c$ of E_F, $\Delta E \simeq k_B T_c$ and

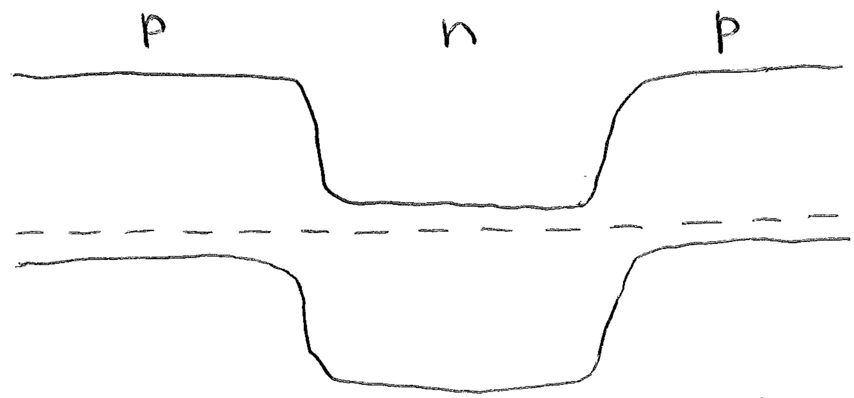
$$\Delta p \simeq \frac{k_B T_c}{v_F}.$$

Thus $$\xi \simeq \frac{\hbar v_F}{k_B T_c} \sim \frac{1}{T_c}.$$

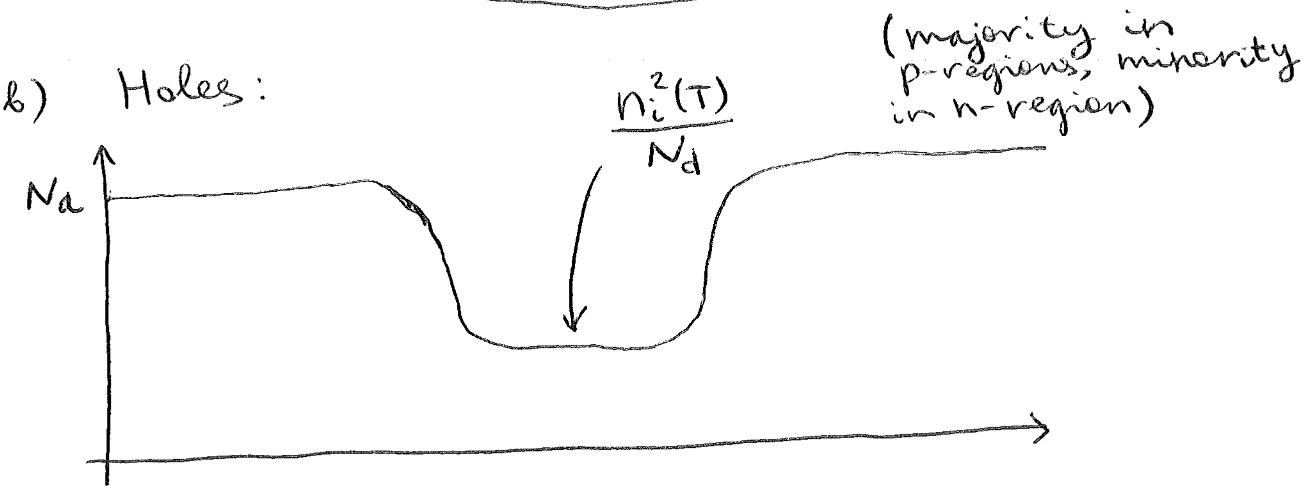
Hence $\xi \uparrow$ as $T_c \downarrow$

2. Please sketch an energy band diagram for a p-n-p junction transistor which is not connected to any external circuits. Assuming that the concentration of impurities is N_a in both p-regions and N_d in the n-region (assume that both junctions are abrupt and that impurities are completely ionized) and that the intrinsic impurity concentration is $n_i(T)$, please sketch minority and majority carrier concentration profiles for the p-n-p transistor.

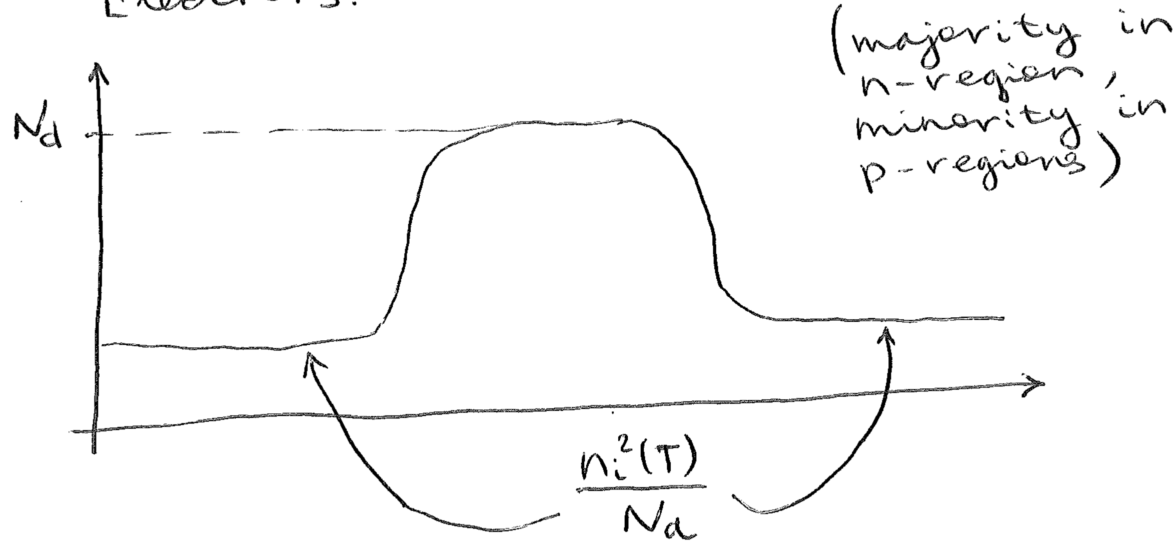
(a)



(b) Holes:



Electrons:



3. a) Please write down the steady-state equations for electron and hole fluxes at a p-n junction which is not connected to any external circuit. Explain the physical origin of each flux contributing to the steady-state balance.

At steady-state,

$J^0_{nr} = J^0_{ng}$ for $\bar{e}$'s, where

J^0_{nr} is the recombination flux ($\bar{e}$'s flow from n-region to p-region & recombine with holes there), and

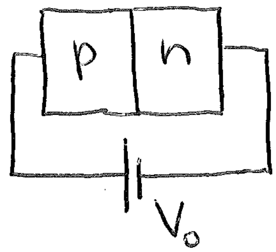
J^0_{ng} is the generation flux ($\bar{e}$'s are created on the p-side by thermal fluctuations & swept to the n-side by the contact potential).

Similarly,

$J^0_{pr} = J^0_{pg}$ for holes.

At steady state recombination and generation fluxes balance each other, separately for $\bar{e}$'s & holes.

b) Now the p-n junction is connected to the battery with a forward bias V_0. Please sketch the circuit diagram (including the battery and the junction), and derive an equation which expresses the total electric current I (carried by both holes and electrons) as a function of the bias voltage V_0 and the zero-voltage fluxes from part (a). Please define each flux in the equation carefully.



Forward bias

With forward bias,

$$J_{ng} = J_{ng}^{0} \quad \text{but} \quad J_{nr} = J_{nr}^{0}\, e^{eV_0/k_BT},$$

all generated $\bar{e}$'s are swept by the junction field anyway

since the barrier is smaller.

The $\bar{e}$ current is $I_n = e(J_{nr} - J_{ng}) =$

$$= e\big(\underbrace{J_{nr}^{0}}_{=J_{ng}^{0} \text{ based on (a)}} e^{eV_0/k_BT} - J_{ng}^{0}\big) = e J_{ng}^{0}\left(e^{eV_0/k_BT} - 1\right).$$

Likewise, the hole current is

$$I_p = e(J_{pr} - J_{pg}) = e\big(\underbrace{J_{pr}^{0}}_{=J_{pg}^{0} \text{ based on (a)}} e^{eV_0/k_BT} - J_{pg}^{0}\big) =$$

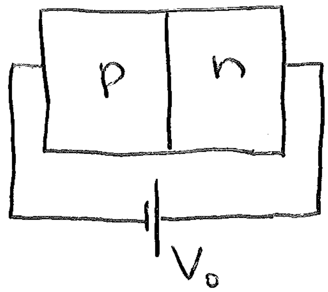
$$= e J_{pg}^{0}\left(e^{eV_0/k_BT} - 1\right).$$

The total current is $I = I_n + I_p =$

$$= \underbrace{e\left(J_{ng}^{0} + J_{pg}^{0}\right)}_{=I_0}\left(e^{eV_0/k_BT} - 1\right).$$

Note that $I \simeq I_0\, e^{eV_0/k_BT}$ if $eV_0 \gg k_BT$

c) Please repeat the procedure in (b) for a reverse bias V_0.



reverse bias

Similarly to (b),

$$I = I_n + I_p = e(J_{ng} - J_{nr}) + e(J_{pg} - J_{pr}) =$$

$$J_{ng} = J_{ng}^{0} \qquad J_{pg} = J_{pg}^{0}$$

$$J_{nr} = J_{nr}^{0} e^{-eV_0/k_BT}$$

$$J_{pr} = J_{pr}^{0} e^{-eV_0/k_BT}$$

$$= e\left(J_{ng}^{0} - J_{nr}^{0} e^{-eV_0/k_BT}\right) +$$

$$+ e\left(J_{pg}^{0} - J_{pr}^{0} e^{-eV_0/k_BT}\right) =$$

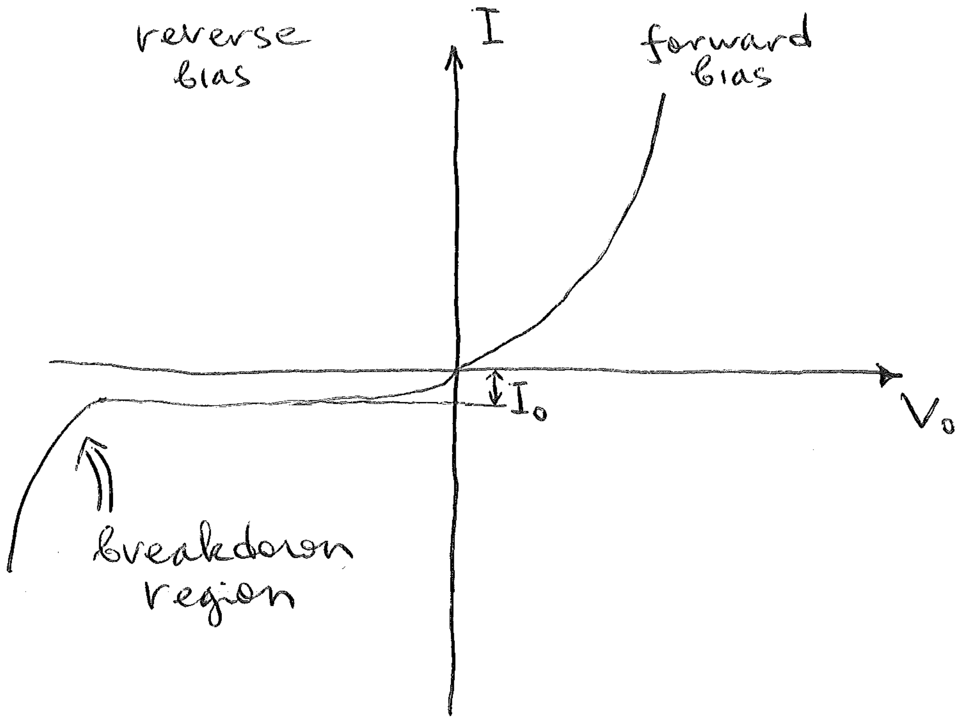
$$= \underbrace{e\left(J_{ng}^{0} + J_{pg}^{0}\right)}_{I_0}\left(1 - e^{-eV_0/k_BT}\right) =$$

$$= I_0\left(1 - e^{-eV_0/k_BT}\right).$$

Note that $I \simeq I_0$ for $eV_0 \gg k_BT \Rightarrow$

$\Rightarrow I_0$ is the saturation current.

d) Please sketch the resulting current-voltage characteristic for the p-n junction, including both forward and reverse bias regions.



4. a) Assume that a 3D metal has a simple cubic lattice with a lattice constant a = 5 Å, and that each atom has one valence electron which becomes a conduction electron in the solid. What is the concentration of conduction electrons per m^3?

In a sc lattice, there is one atom (and thus 1 conduction $\bar{e}$) per unit cell. Therefore

$$n = \frac{1\bar{e}}{(5\text{Å})^3} = 8 \times 10^{-3}\ \text{Å}^{-3} = \underline{\underline{8 \times 10^{27}\ m^{-3}}}.$$

b) In the framework of a free electron model, derive the formula expressing the Fermi energy E_F as a function of electron concentration n. What is the Fermi energy for the metal from part (a) (in eV)? Recall that electron mass is $m=9.1\times10^{-31}$ kg and Planck's constant is $\hbar=1.05\times10^{-34}$ J·s.

Recall that

$$\underbrace{N(k)}_{\#\text{ states up to }k} = \frac{\frac{4\pi}{3}k^3}{\left(\frac{2\pi}{L}\right)^3},$$

where L is the sample dimension.

Then

$$\underbrace{N(k_F)}_{\frac{N}{2}\text{ due to Spin}} = \frac{\frac{4\pi}{3}k_F^3}{\frac{8\pi^3}{V}}, \qquad V=L^3$$

$$k_F^3 = \frac{N}{2V}6\pi^2 = 3\pi^2 \underbrace{n}_{\bar{e}\text{ concentr'n}}$$

Thus

$$E_F = \frac{\hbar^2}{2m}k_F^2 = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$$

For the metal in (a),

$$E_F = \frac{(1.05\times10^{-34})^2\ J^2\cdot s^2}{2\times9.1\times10^{-31}\,kg}\left(3\pi^2\times8\times10^{27}\,m^{-3}\right)^{2/3} \simeq$$

$$\simeq 2.32\times10^{-19}\,J = 1.45\ eV.$$

$1eV = 1.6\times10^{-19}$ J

5. a) The average energy of a quantum-mechanical oscillator at equilibrium with temperature T is given by

   $\langle\varepsilon\rangle = \hbar\omega/(\exp(\hbar\omega/ k_B T) - 1)$,

where $\hbar\omega$ is the energy difference between adjacent energy levels of an isolated oscillator. In the framework of the Einstein theory of specific heat, please write down the internal energy per unit volume of the solid. The concentration of atoms in a solid is n.

In Einstein theory, there is only one ω, ω_E. Then the internal energy per unit volume is simply

$$E = 3n \frac{\hbar\omega_E}{e^{\hbar\omega_E/k_BT} - 1}$$

atom concentration

degrees of freedom for each atom

b) Derive the formula for the specific heat C_V in the Einstein model. What is the behavior of the specific heat as $T \to 0$ (such that $k_B T \ll \hbar\omega$)?

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V \quad , \text{ so}$$

$$C_V = 3n\hbar\omega_E \left(-\frac{1}{(e^{\hbar\omega_E/k_BT}-1)^2}\right) e^{\hbar\omega_E/k_BT} \frac{\hbar\omega_E}{k_B}\left(-\frac{1}{T^2}\right) =$$

$$= 3nk_B \left(\frac{\hbar\omega_E}{k_BT}\right)^2 \frac{e^{\hbar\omega_E/k_BT}}{(e^{\hbar\omega_E/k_BT}-1)^2} .$$

This is per unit volume; to find C_V per mole, replace n with N_A. (Avogadro #)

Then

$$C_V = 3\underbrace{R}_{N_A k_B} \left(\frac{\hbar\omega_E}{k_BT}\right)^2 \frac{e^{\hbar\omega_E/k_BT}}{(e^{\hbar\omega_E/k_BT}-1)^2} .$$

As $T \to 0$, $\hbar\omega_E \gg k_BT$ and

$$C_V \simeq 3R\left(\frac{\hbar\omega_E}{k_BT}\right)^2 e^{-\hbar\omega_E/k_BT} .$$

Thus C_V approaches $T=0$ exponentially.

6. a) Imagine a 1D random walker which starts from $x_0=0$ and takes a step of length L to the right or to the left with equal probabilities (the probability to stay in the same position is 0). Derive the average displacement, $\langle x_N \rangle$, after N steps for an ensemble of such walkers.

The walker steps to the right with probability $1/2$, and to the left with prob. $1/2$. Introduce a random variable k:

$$k = \begin{cases} +1 , & p = 1/2 \\ -1 , & p = 1/2 \end{cases}$$

Then $x_N = x_{N-1} + k \underbrace{L}_{\text{step length}}$.

The average displacement is

$$\langle x_N \rangle = \langle x_{N-1} \rangle + L \underbrace{\langle k \rangle}_{\text{0, equal prob. to be } \pm 1} = \langle x_{N-1} \rangle .$$

But this means that there is no change on average:

$$\langle x_N \rangle = \langle x_{N-1} \rangle = \ldots = \langle x_0 \rangle = \underbrace{x_0}_{\text{all particles start from } x_0} = 0 .$$

Thus $\underline{\underline{\langle x_N \rangle = 0}}$.

b) Derive the average squared displacement, $\langle x_N^2 \rangle$, after N steps for the ensemble of random walkers from part (a). What is the dependence of $\langle x_N^2 \rangle$ on the number of steps N?

Using $x_N = x_{N-1} + kL$ as in part (a),

$$\langle x_N^2 \rangle = \langle (x_{N-1} + kL)^2 \rangle = \langle x_{N-1}^2 \rangle +$$

$$+ 2L \underbrace{\langle x_{N-1} k \rangle}_{= \langle x_{N-1} \rangle \langle k \rangle = 0} + L^2 \underbrace{\langle k^2 \rangle}_{= 1} =$$

$\langle x_{N-1} \rangle \langle k \rangle = 0$: After N-1 steps, equal prob. to go left or right, so will cancel out

$\langle k^2 \rangle = 1$, k^2 same for steps to the left & to the right

$$= \langle x_{N-1}^2 \rangle + L^2 .$$

So,

$$\langle x_N^2 \rangle = \langle x_{N-1}^2 \rangle + L^2 =$$

$$= \langle x_{N-2}^2 \rangle + 2L^2 = \ldots =$$

$$= \underbrace{\langle x_0^2 \rangle}_{= 0} + NL^2 .$$

$x_0^2 = 0$, all walkers start at origin

$$\langle x_N^2 \rangle = NL^2 \underline{\underline{\sim N}}$$

, grows linearly with N.

7. a) What is the average energy of an electron in a 3D Fermi gas at T=0? Please express your answer in terms of the Fermi energy E_F. Hint: you may want to average electron energy over a sphere in k-space.

$$E_k = \frac{\hbar^2 k^2}{2m}$$

$$\underbrace{\langle E_k \rangle}_{\text{average in k-space}} = \frac{\hbar^2}{2m} \frac{\int_0^{k_F} dk\, k^4}{\int_0^{k_F} dk\, k^2} = \frac{\hbar^2}{2m} \frac{3}{5} k_F^2 =$$

$$= \frac{3}{5} E_F.$$

b) Using results from part (a), find the average electron energy in a 3D solid with E_F = 6 eV.

$$\langle E_k \rangle = \frac{3}{5} 6\text{ eV} = 3.6\text{ eV}.$$

8. Consider a 2D array of atoms arranged in a regular square lattice, with a lattice constant of 4 Å.

a) Assuming that the phonon dispersion is well described by the Debye approximation (ω = ck), find the phonon density of states g(ω) for this 2D solid.

In 2D,

$$N(k) = \frac{\pi k^2}{\left(\frac{2\pi}{L}\right)^2} = \frac{Ak^2}{4\pi},$$ where $A = L^2$ is the area.

Since $\omega = ck$,

$$N(\omega) = \frac{A\omega^2}{4\pi c^2}.$$

Finally, $$\underbrace{D(\omega)}_{\equiv\, g(\omega)} = \underset{\substack{\uparrow \\ \text{\# modes in a 2D solid}}}{2}\,\frac{dN}{d\omega} = \frac{A\omega}{\pi c^2}.$$

b) If the speed of sound is c = 10^3 m/s, what is the Debye temperature of this 2D solid? Hint: use the density of states from part (a). You may need these constants: $\hbar=1.05\times10^{-34}$ J·s, $k_B=1.38\times10^{-23}$ J/K.

The cutoff freq. is defined by

$$\int_0^{\omega_D} g(\omega)\,d\omega = 2N \quad \text{in 2D, or}$$

$$\frac{A\omega_D^2}{4\pi c^2} = N \Rightarrow \omega_D = \left[4\pi c^2\left(\frac{N}{A}\right)\right]^{1/2}.$$

Debye freq.

atoms per unit area

$$\frac{N}{A} = \frac{1}{16\,\mathring{A}^2} = 6.25\times10^{18}\ m^{-2}.$$

So, $\omega_D = 2\times10^3\ \frac{m}{s}\sqrt{\pi}\sqrt{6.25\times10^{18}\ m^{-2}} \simeq$

$$\simeq 8.86\times10^{12}\ s^{-1}.$$

Now, $k_B\Theta_D = \hbar\omega_D$ gives

$$\Theta_D = \frac{1.05\times10^{-34}\ J\cdot s}{1.38\times10^{-23}\ J/K}\times 8.86\times10^{12}\ s^{-1} \simeq$$

$$\simeq 67.4\ K.$$

9. a) An incident monochromatic X-ray beam with wavelength $\lambda = 1.5$ Å is reflected from the (111) plane in a 3D solid with a Bragg angle of 22° for the n=1 reflection. Please compute the distance (in Å) between adjacent (111) planes.

Bragg's law

$$\underbrace{n}_{n=1 \text{ here}}\lambda = 2d\sin\theta$$

In our case,

$$d_{111} = \frac{\lambda}{2\sin\theta} = \frac{1.5\text{ Å}}{2\sin 22°} \simeq 2.0\text{ Å}$$

b) Assuming that the solid has an fcc lattice, use the result from part (a) to compute the lattice constant (in Å).

fcc lattice:

$$d_{111} = \frac{a}{\sqrt{1^2+1^2+1^2}} \Rightarrow a = d_{111}\sqrt{3} =$$

$$d_{hkl} = \frac{a}{\sqrt{h^2+k^2+l^2}}$$

$$= 2.0\text{ Å} \times \sqrt{3} \simeq 3.46\text{ Å}.$$

10. Please describe recent accomplishments and future challenges in the field of photonic crystals. Please use a minimum of 4 sentences in your response.

Photonic crystals are materials carefully designed to create a photonic band gap - a range of wavelengths of light blocked by the material. Despite initial difficulties, the field of photonic crystals has thrived. Among recent achievements are a new kind of optical fiber and nanoscopic lasers. Under active development are photonic integrated circuits, in which 2D films can be patterned to make various optical devices. Such optical circuits would represent a triumph of optoelectronic miniaturization.