Physics 406, Spring 2013

Final Exam
May 10, 2013

Name solutions

The ten problems are worth 10 points each.

Problem	Score
1	10
2	10
3	10
4	10
5	10
6	10
7	10
8	10
9	10
10	10

1. a) Derive the relationship between H_C (0), the critical field at zero temperature, and the critical temperature T_C for Type I superconductors. Hint: estimate the demagnetization energy and equate it with the energy carried by superconducting electrons at concentration n_{eff}.

As shown in class,

$$\Delta E = \frac{1}{2}\mu_0 H_c^2(0)$$

$\leftarrow T=0$

On the other hand,

$$\Delta E \simeq n_{eff} k_B T_c \ , \text{ where } n_{eff} \simeq n \frac{k_B T_c}{E_F}$$

n: total concentr'n of conduction $\bar{e}$'s

n_{eff}: $\bar{e}$'s involved in sc

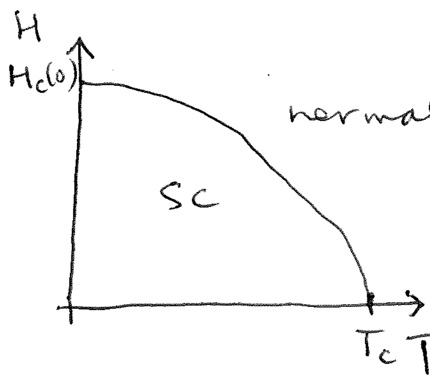
So,

$$\frac{1}{2}\mu_0 H_c^2(0) = n \frac{(k_B T_c)^2}{E_F},$$

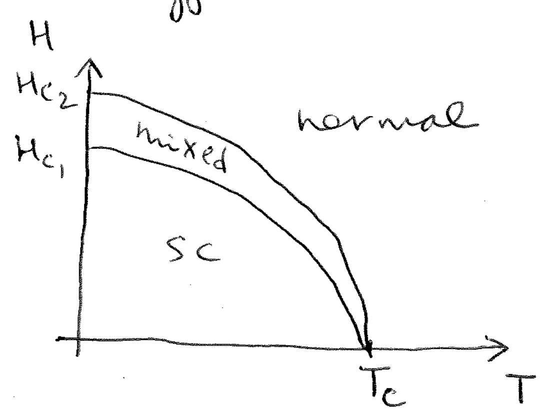
$$\underline{\underline{H_c(0) = \left(\frac{2n}{\mu_0 E_F}\right)^{1/2} k_B T_c}}.$$

b) Use phase diagrams to explain essential phenomenological differences between Type I and Type II superconductors.

Type I

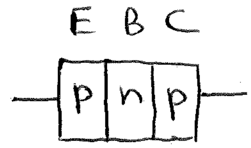


Type II

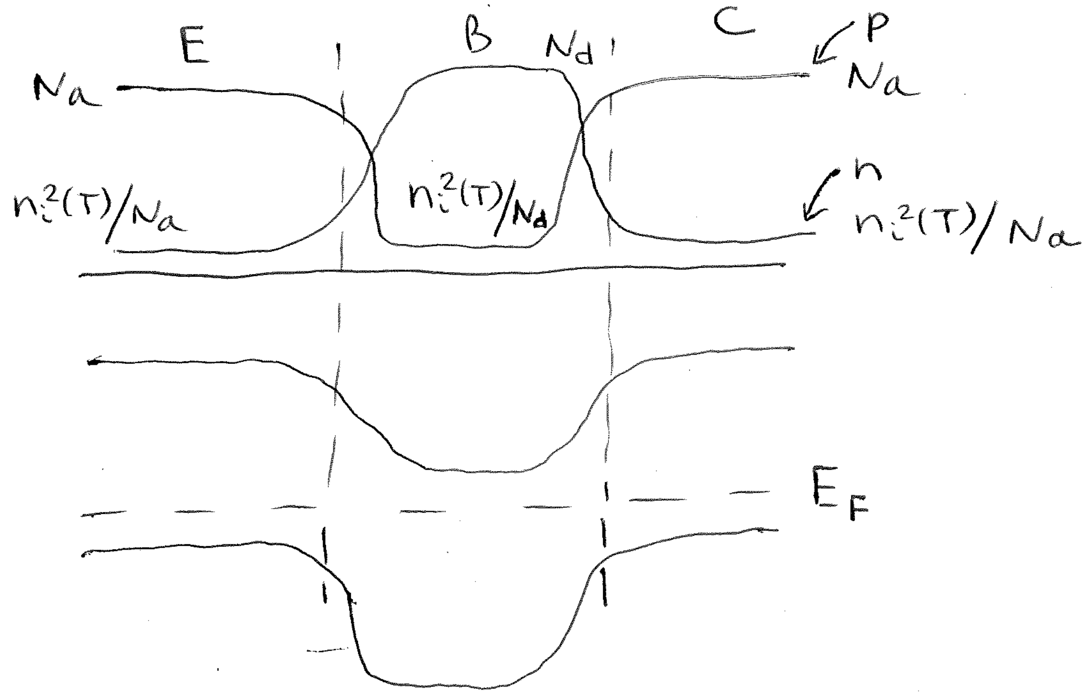


Type II sc have a mixed phase in which vortices in normal state are surrounded by sc state.

2. Please sketch an energy band diagram for a p-n-p junction transistor which is not connected to any external circuits. Assuming that the concentration of impurities is N_a in the p-region and N_d in both n-regions (assume that both junctions are abrupt and that impurities are completely ionized) and that the intrinsic impurity concentration is $n_i(T)$, sketch minority and majority carrier concentration profiles across the p-n-p transistor. Also, sketch the band structure diagram for a p-n-p junction transistor at equilibrium, in the absence of external voltage. Clearly label emitter, base and collector regions on both diagrams.



p : N_a
n : N_d



3. a) Please write down the steady-state equations for electron and hole fluxes across a p-n junction which is not connected to an external circuit. Explain the physical origin of each flux contributing to the steady-state balance.

At steady state,

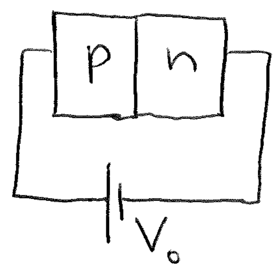
$$J_{nr}^{0} = J_{ng}^{0} \text{ for } \bar{e}\text{'s},$$

where J_{nr}^{0} is the recombination flux ($\bar{e}$'s flow from n to p region and recombine with holes), and J_{ng}^{0} is the generation flux ($\bar{e}$'s are created on the p side & swept to the n side by the electric field at the junction).

Similarly, $J_{pr}^{0} = J_{pg}^{0}$ for holes.

At ss recombination & generation fluxes are balanced separately for $\bar{e}$'s & holes.

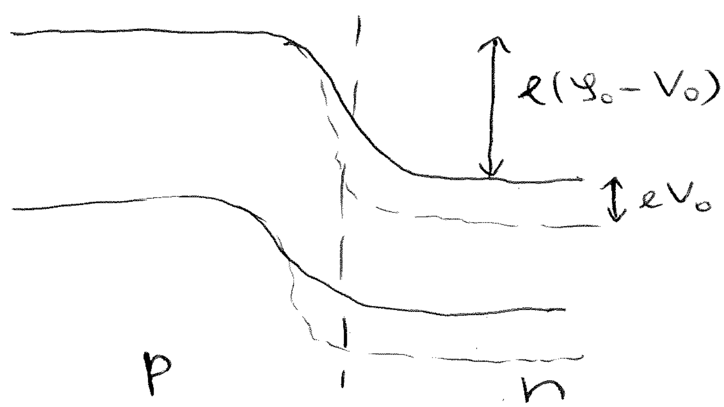
b) Now the p-n junction is connected to the battery with a forward bias V_0. Please sketch the circuit diagram (including the battery and the junction), the band structure diagram, and derive an equation which expresses the total electric current *I* (carried by both holes and electrons) as a function of the bias voltage V_0 and the zero-voltage fluxes from part (a). Please define each flux in the equation carefully.



forward bias

Here, $J_{ng} = J_{ng}^0$ but $J_{nr} = J_{nr}^0 e^{eV_0/k_BT}$ since the barrier is smaller

all $\bar{e}$'s are swept from p to n anyway



The $\bar{e}$ current is $I_n = e(J_{nr} - J_{ng}) =$

$$= e\left(J_{nr}^0 e^{eV_0/k_BT} - J_{ng}^0\right) = e J_{ng}^0 \left(e^{eV_0/k_BT} - 1\right).$$

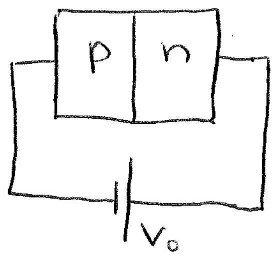
$\parallel J_{ng}^0$ from (a)

Likewise, $I_p = e J_{pg}^0 \left(e^{eV_0/k_BT} - 1\right)$.

hole current

$$I = I_n + I_p = \underbrace{e\left(J_{ng}^0 + J_{pg}^0\right)}_{\parallel I_0}\left(e^{eV_0/k_BT} - 1\right) \simeq I_0 e^{eV_0/k_BT} \text{ if } eV_0 \gg k_BT$$

c) Please repeat the procedure in (b) for a reverse bias V_0. Do not forget to sketch both the circuit and the band structure diagrams.



reverse bias

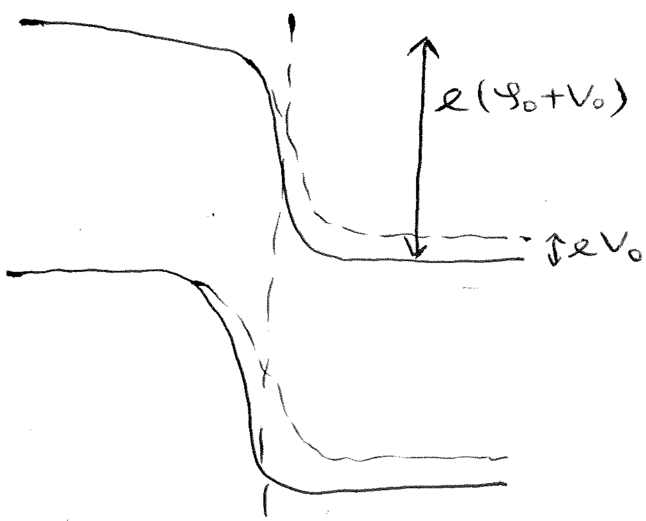
as in (b), $I = I_n + I_p = e(J_{ng} - J_{nr}) +$

$+ e(J_{pg} - J_{pr}) =$

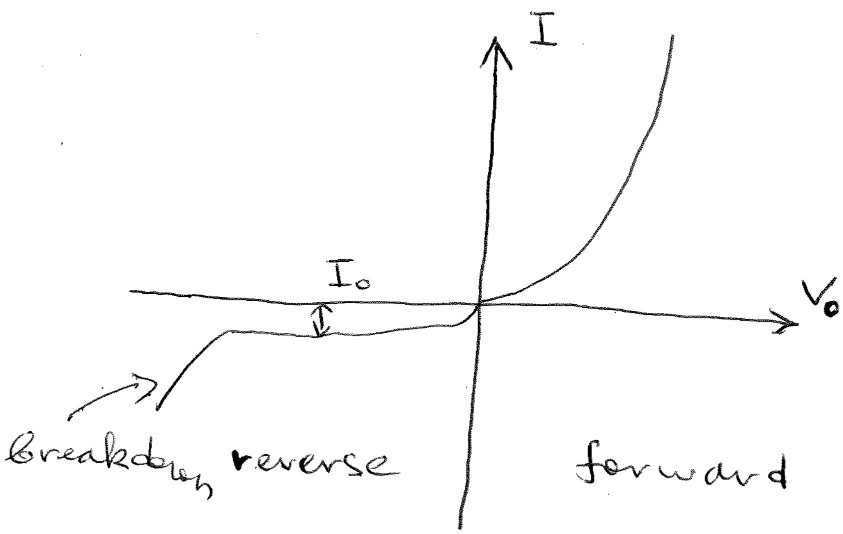
$e(J_{ng}^0 - J_{nr}^0 e^{-eV_0/k_BT}) + e(J_{pg}^0 - J_{pr}^0 e^{-eV_0/k_BT}) =$

$= \underbrace{e(J_{ng}^0 + J_{pg}^0)}_{I_0 \text{ from (b)}}(1 - e^{-eV_0/k_BT}) \simeq I_0$ if $eV_0 \gg k_BT$

Thus I_0 is the saturation current.



d) Please sketch the resulting current-voltage characteristic for the p-n junction, including both forward and reverse bias regions. Explain what happens when the reverse bias is so large that the breakdown of the p-n junction occurs.



$$I = I_0 \left(e^{eV_0/k_BT} - 1 \right)$$

$V_0 > 0$ forward bias

$V_0 < 0$ reverse bias

Reverse current increases rapidly for large negative V_0.

Two reasons:

(a) avalanche breakdown ($\bar{e}$-hole excitation)

(b) Zener (QM) breakdown due to tunneling across a very thin barrier

4. a) Assume that a 3D metal has a bcc lattice with a lattice constant a = 5 Å, and that each atom has one valence electron which becomes a conduction electron in the solid. What is the concentration of conduction electrons per m^3?

bcc lattice: 2 atoms per unit cell

1 atom = 1 conduction $\bar{e}$

Thus $n = \dfrac{2}{(5\,\text{Å})^3} = 0.016\ \text{Å}^{-3} =$

$= 1.6 \times 10^{28}\ m^{-3}$.

b) Using the free electron model, derive the formula expressing the Fermi energy E_F as a function of electron concentration n. What is the Fermi energy for the 3D metal from part (a) (in eV)? Recall that free electron mass is $m=9.1\times10^{-31}$ kg and Planck's constant is $\hbar=1.05\times10^{-34}$ J·s.

Recall that

$$N(k) = \frac{\frac{4}{3}\pi k^3}{\left(\frac{2\pi}{L}\right)^3},$$

$L^3 = V$ is the sample volume

Then $$N(k_F) = \frac{4}{3}\pi \frac{1}{8\pi^3} V k_F^3 =$$

$$= \frac{1}{6\pi^2} V k_F^3, \text{ or}$$

$$N = \frac{1}{3\pi^2} V k_F^3. \quad (\times 2 \text{ due to spin})$$

Then $$k_F = \Big(3\pi^2 \underbrace{\frac{N}{V}}_{n - \bar{e}\text{ conc'n}}\Big)^{1/3}$$

Finally, $$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}.$$

With n from (a),

$$E_F \simeq 1.07 \text{ eV}.$$

5. a) Derive the expression for the density of modes g(ω) in a wave traveling in 1D continuous medium of length L. Assume that dispersion is linear: ω=ck (where c is the velocity of sound), and that periodic boundary conditions apply.

In 1D, "area" of sphere w/radius k

$$N(k) = \frac{2k}{2\pi/L} = \frac{Lk}{\pi}.$$

$$\omega = ck \Rightarrow N(\omega) = \frac{L\omega}{\pi c}.$$

$$D(\omega) = \frac{dN}{d\omega} = \underline{\underline{\frac{L}{\pi c}}}.$$

$$D(\omega) \equiv g(\omega)$$

b) Repeat the derivation in (a) for 3D continuous medium of volume V. Discuss the dependence of g(ω) on ω in 1D vs. 3D cases?

In 3D,

$$N(k) = \frac{\frac{4}{3}\pi k^3}{\left(\frac{2\pi}{L}\right)^3} = \frac{1}{6\pi^2} V k^3,$$

where $V = L^3$.

Then $N(\omega) = \frac{1}{6\pi^2} \frac{V\omega^3}{c^3}$, and

$$\underset{\text{"}g(\omega)\text{"}}{D(\omega)} = \underbrace{3}_{\substack{1L+2T \\ \text{modes}}} \frac{dN}{d\omega} = \underline{\underline{\frac{3V}{2\pi^2} \frac{\omega^2}{c^3}}}$$

Note that $g(\omega) \sim \omega^2$, unlike the 1D case where $g(\omega)$ is indep. of ω.

c) Using the expression for $g(\omega)$ from part (b), derive the expression for Debye frequency in a 3D solid. What is its dependence on n, the concentration of atoms in the solid?

ω_D is defined by

$$\int_0^{\omega_D} d\omega\, g(\omega) = 3 N_A . \quad \text{\# atoms}$$

with $g(\omega)$ from (b),

$$\frac{3V}{2\pi^2} \frac{1}{c^3} \int_0^{\omega_D} d\omega\, \omega^2 = 3 N_A, \text{ or}$$

$$\frac{V}{2\pi^2 c^3} \omega_D^3 = 3 N_A$$

$$\Downarrow$$

$$\omega_D = c \left(6\pi^2 n\right)^{1/3}, \quad n = \frac{N_A}{V}.$$

atom concentr'n

Note that $\omega_D \sim n^{1/3}$

6. a) Consider a 1D random walker which starts from $x_0=0$ and takes a step of fixed length L to the right or to the left with equal probabilities $p_L = p_R$ (the probability to stay in the same position is 0). Derive the average displacement, $<x_N>$, after N steps for an ensemble of such walkers.

Introduce a random variable

$$k = \begin{cases} +1, & p_R = \frac{1}{2} \\ -1, & p_L = \frac{1}{2} \end{cases}$$

step length

Then $x_N = x_{N-1} + kL$

$\langle x_N \rangle = \langle x_{N-1} \rangle + \langle k \rangle L$, where

$\langle k \rangle = \frac{1}{2}(+1) + \frac{1}{2}(-1) = 0$.

Thus $\langle x_N \rangle = \langle x_{N-1} \rangle \Rightarrow \langle x_N \rangle = x_0 = 0$.

The CM of all walkers never moves.

b) Derive the average squared displacement, $\langle x_N^2 \rangle$, after N steps for the ensemble of unbiased random walkers from part (a) ($p_L = p_R$). What is the dependence of $\langle x_N^2 \rangle$ on the number of steps N?

Again, use $x_N = x_{N-1} + kL$:

$$\langle x_N^2 \rangle = \langle x_{N-1}^2 \rangle + 2L \langle x_{N-1} k \rangle + L^2 \langle k^2 \rangle =$$

$$\langle k^2 \rangle = \tfrac{1}{2}(+1) + \tfrac{1}{2}(+1) = 1$$

$$\langle x_{N-1} k \rangle = \langle x_{N-1} \rangle \langle k \rangle = 0$$

equally likely to go left or right after $N-1$ steps

$$= \langle x_{N-1}^2 \rangle + L^2.$$

Then recursively

$$\langle x_N^2 \rangle = \underbrace{\langle x_0^2 \rangle}_{=0\text{, all walkers start at } x_0 = 0} + NL^2 = NL^2.$$

Thus $\langle x_N^2 \rangle \sim N$, grows linearly w/ N.

7. a) What is the average energy of an electron in a 3D Fermi gas at T=0 K? Please express your answer in terms of the Fermi energy E_F. Hint: you may want to average the electron energy over a sphere in k-space. Find the average electron energy at T=0 K for a 3D solid with $E_F = 6.0$ eV.

$$E_k = \frac{\hbar^2 k^2}{2m}$$

$$\langle E_k \rangle = \frac{\hbar^2}{2m} \frac{\int_0^{k_F} dk\, k^4}{\int_0^{k_F} dk\, k^2} = \frac{\hbar^2}{2m} \frac{3}{5} k_F^2 =$$

$$= \frac{3}{5} E_F .$$

If $E_F = 6.0$ eV, $\langle E_k \rangle = 3.6$ eV.

b) Write down the Fermi-Dirac distribution and estimate the probability of an electron to be 0.1 eV above the Fermi level in a solid from part (a), at T=300 K. Recall that $k_B=1.38 \times 10^{-23}$ J/K = 8.6×10^{-5} eV/K.

$$f(E) = \frac{1}{e^{(E-E_F)/k_BT} + 1}$$

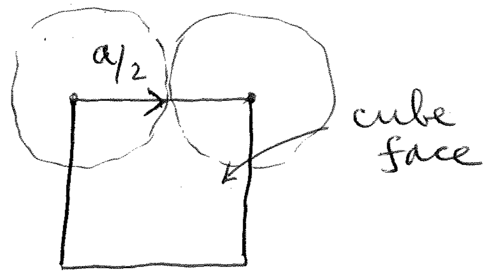
$E_F = 6.0$ eV, $T = 300$ K,

$E = E_F + 0.1$ eV :

$$f(E) \simeq \underline{\underline{0.02}}$$

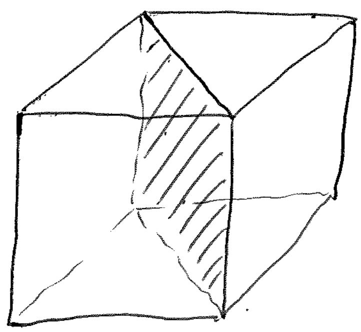
8. Compute the radius of spheres in a close-packed arrangement on sc, bcc and fcc lattices with cube edge length a. Note: in a close-packed arrangement, sphere radius cannot be increased any further without steric overlap between some of the adjacent spheres.

sc:

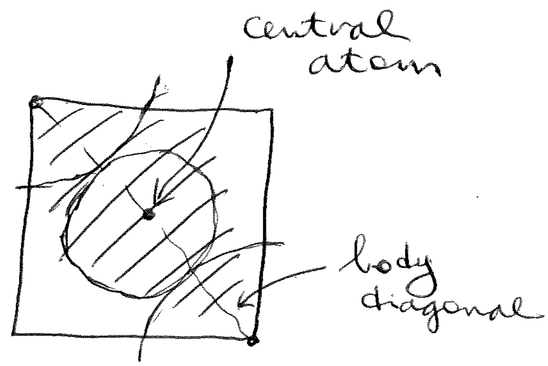


$$r_{sc} = \frac{a}{2}$$

bcc:

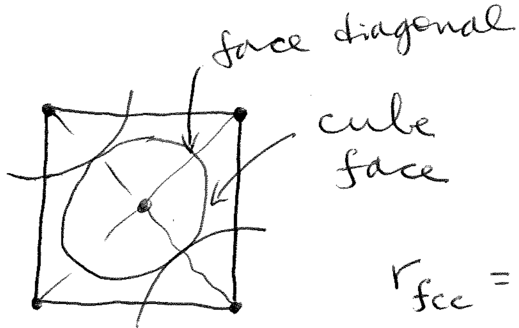


$\Rightarrow$



$$r_{bcc} = \frac{a\sqrt{3}}{4} < r_{sc}$$

fcc:



$$r_{fcc} = \frac{a\sqrt{2}}{4} < r_{bcc}$$

9. a) An incident monochromatic X-ray beam with wavelength λ = 1.9 Å is reflected from the (111) plane in a 3D solid with a Bragg angle of 32° for the n=1 reflection. Please compute the distance (in Å) between adjacent (111) planes.

Bragg's law:

$$\underbrace{n}_{=1 \text{ here}}\lambda = 2d\sin\theta .$$

Here, $d_{111} = \dfrac{\lambda}{2\sin\theta} = \dfrac{1.9\ \text{Å}}{2\sin(32°)} \simeq$

$$\simeq 1.79\ \text{Å} .$$

b) Assuming that the solid has an fcc lattice, use the result from part (a) to compute the lattice constant (in Å).

Recall that

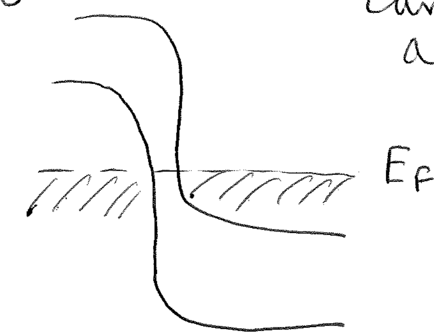
$d_{hkl} = \dfrac{a}{\sqrt{h^2+k^2+l^2}}$ in the fcc lattice $\Rightarrow d_{111} = \dfrac{a}{\sqrt{3}}$, or

$$a = 1.79\ \text{Å} \times \sqrt{3} \simeq 3.1\ \text{Å} .$$

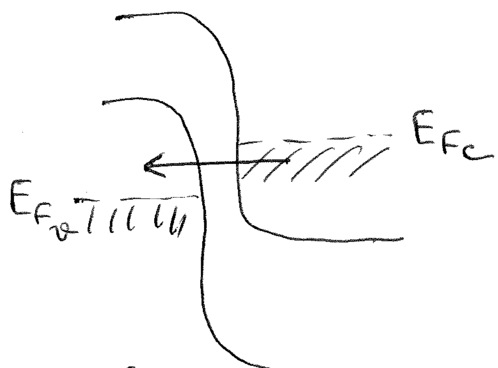
10. Please describe the basic idea behind a tunnel diode. Use band structure sketches to justify your answers. Also, sketch the current-voltage characteristic for this device and explain qualitative differences between different regions under forward bias. Discuss similarities and differences between a tunnel diode and a regular p-n junction.

Heavily doped p-n junction: QM tunneling dominates, not minority carrier diffusion as in ordinary p-n junctions

No bias:

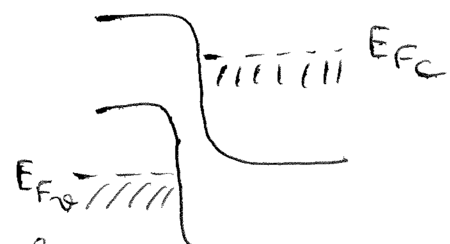


Small forward bias:



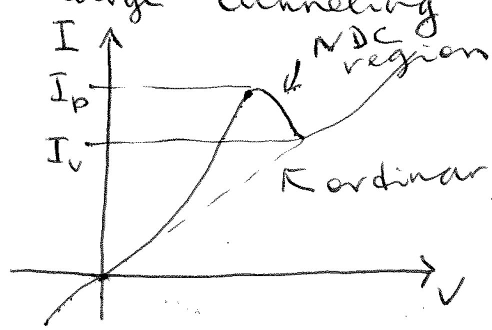
tunneling

Large forward bias:



No tunneling, bands uncrossed

Reverse bias: large tunneling current, Zener breakdown



$\frac{I_p}{I_v}$ up to ≈ 15