Physics 313
Midterm Exam

October 27, 2015

The exam is 80 minutes in length.

There is a total of five problems. Each problem is worth 20 points regardless of its length or number of parts.

You may refer ONLY to a single double-sided sheet of paper with notes (US Letter size) that you brought with you. You may also use a calculator.

Do not forget to write your name on the first page!

Good luck!

Problem 1

One spaceship flies away from Earth at $0.7c$, where c is the speed of light. The other one flies toward Earth at $0.9c$. The length of the first spaceship (a colony cylinder world) is $L_1 = 5000\ m$ in its own frame of reference. The length of the second spaceship (an advanced warcraft) is $L_2 = 120\ m$ in its own frame of reference.

(a) Assuming that Earth is stationary, find the lengths of the first and the second ships in the Earth reference frame.

$$\gamma_{0.7c} = \left(\sqrt{1-(0.7)^2}\right)^{-1} = 1.40$$

$$\gamma_{0.9c} = \left(\sqrt{1-(0.9)^2}\right)^{-1} \simeq 2.29$$

Thus

$$\left.\begin{aligned} L_1^{Earth} &= \frac{L_1}{\gamma_{0.7c}} \simeq 3570\ m \\ L_2^{Earth} &= \frac{L_2}{\gamma_{0.9c}} \simeq 52.4\ m \end{aligned}\right\} \text{length contraction}$$

(b) A time interval of $\Delta\tau = 1$ second is measured independently on-board both ships. How much time passes for the Earth-bound observer in each case?

Likewise,

$$\left.\begin{aligned} \Delta\tau_1^{Earth} &= \gamma_{0.7c}\,\Delta\tau \simeq 1.40\ s \\ \Delta\tau_2^{Earth} &= \gamma_{0.9c}\,\Delta\tau \simeq 2.29\ s \end{aligned}\right\} \text{time dilation}$$

(c) What is the velocity of the second ship in the frame of reference attached to the first ship? What is the length of the second ship in that frame?

Use $$u' = \frac{u - v}{1 - \frac{uv}{c^2}},$$

with $u = -0.9c$ and $v = 0.7c$.
$\uparrow$ toward Earth

Then $$u' = \frac{-0.9c - 0.7c}{1 - \frac{(-0.9c)(0.7c)}{c^2}} \approx -0.98c.$$

Finally,

$$\gamma_{0.98c} = \left(\sqrt{1-(0.98)^2}\right)^{-1} \approx 5.03.$$

Then $$L\left(\begin{matrix}\text{ship 2} \\ \text{in ship 1} \\ \text{frame}\end{matrix}\right) = \frac{L_2}{\gamma_{0.98c}} \approx 23.9 \text{ m}.$$

Problem 2

A particle of mass m moves at velocity u with respect to the stationary observer when it instantaneously divides into two particles. Particle 1 has mass $m_1 < m$ and velocity $u_1 = u$. Assuming that the situation is one-dimensional, what are the mass m_2 and velocity u_2 of particle 2?

Momentum conservation:

$$\gamma_u m u = \gamma_{u_1} m_1 u_1 + \gamma_{u_2} m_2 u_2$$

Energy conservation:

$$\gamma_u m c^2 = \gamma_{u_1} m_1 c^2 + \gamma_{u_2} m_2 c^2$$

$$\begin{cases} \gamma_{u_2} m_2 u_2 = \gamma_u m u - \gamma_{u_1} m_1 u_1, \\ \gamma_{u_2} m_2 = \gamma_u m - \gamma_{u_1} m_1 \end{cases}$$

Then $$u_2 = \frac{\gamma_u m u - \gamma_{u_1} m_1 u_1}{\gamma_u m - \gamma_{u_1} m_1} =$$

$$\underset{u=u_1}{=} u \frac{m - m_1}{m - m_1} = \underline{\underline{u}}.$$

Finally, $$m_2 = \frac{\gamma_u m - \gamma_{u_1} m_1}{\gamma_{u_2}} =$$

$$= \underline{\underline{m - m_1}}.$$

So the particle divides in two but both new particles continue with the same velocity.

Problem 3

In photoelectric effect, it was found that light with $\lambda_1 = 600$ nm wavelength ejects electrons with a maximum speed of 0 m/s from a metal plate (i.e. the electron is liberated from the metal with no kinetic energy). What will the maximum possible speed of electrons be if light with $\lambda_2 = 300$ nm wavelength strikes the same metal plate? If you use the non-relativistic expression for electron's energy, please justify its applicability!

Recall that for photons,

$$E = pc = hf = \frac{hc}{\lambda}.$$

Thus in photoelectric effect,

$$KE_{max} = \frac{hc}{\lambda} - \underbrace{\varphi}_{\text{work function}}$$

Thus

$$\varphi = \underbrace{\frac{hc}{\lambda_1}}_{KE_{max,1}=0} \simeq 3.3 \times 10^{-19}\ J.$$

Then

$$KE_{max,2} = \frac{hc}{\lambda_2} - \varphi \simeq 3.3 \times 10^{-19}\ J.$$

Using

$$KE_{max,2} = \frac{m_e v_{max}^2}{2},$$

we obtain:

$$v_{max} = \sqrt{\frac{2\, KE_{max,2}}{m_e}} \simeq \underline{\underline{8.5 \times 10^5 \frac{m}{s}}}.$$

$v_{max} << c$, so the non-relativistic expression is OK.

Problem 4

Imagine that the wave function is given by

$$\psi(x) = Ae^{-x/L}, \text{ for } x \geq 0$$

where A is the normalization constant.

(a) Find A.

$$\int_0^\infty dx\, |\Psi(x)|^2 = 1 \Rightarrow |A|^2 \int_0^\infty dx\, e^{-\frac{2x}{L}} = |A|^2 \frac{L}{2} = 1,$$

$$\text{or } A = \sqrt{\frac{2}{L}}.$$

(b) Find $\langle x \rangle$, $\langle x^2 \rangle$. and the RMS of x, σ_x. For the last part, first express σ_x in terms of $\langle x \rangle$ and $\langle x^2 \rangle$.

$$\langle x \rangle = \int_0^\infty dx\, x |\Psi(x)|^2 = \frac{2}{L} \int_0^\infty dx\, x\, e^{-\frac{2x}{L}} =$$

$$= \frac{2}{L} \left[\cancel{x \left(-\frac{L}{2}\right) e^{-\frac{2x}{L}} \Big|_0^\infty} - \int_0^\infty dx \left(-\frac{L}{2}\right) e^{-\frac{2x}{L}} \right] =$$

$$= \int_0^\infty dx\, e^{-\frac{2x}{L}} = \frac{L}{2}.$$

Likewise, $\langle x^2 \rangle = \frac{2}{L} \int_0^\infty dx\, x^2 e^{-\frac{2x}{L}} =$

$$= \frac{2}{L} \left[\cancel{x^2 \left(-\frac{L}{2}\right) e^{-\frac{2x}{L}} \Big|_0^\infty} - \int_0^\infty dx\, (2x) \left(-\frac{L}{2}\right) e^{-\frac{2x}{L}} \right] =$$

$$= 2 \underbrace{\int_0^\infty dx\, x\, e^{-\frac{2x}{L}}}_{\left(\frac{L}{2}\right)^2} = \frac{L^2}{2}.$$

Finally, $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 =$

$$= \frac{L^2}{2} - \frac{L^2}{4} = \frac{L^2}{4} \Rightarrow \sigma_x = \frac{L}{2}.$$

Problem 5

Recall that for the harmonic oscillator with $U(x) = \frac{1}{2}\kappa x^2$ and a particle of mass m, the ground-state wave function is given by

$$\psi(x) = \left(\frac{m\kappa}{\pi^2\hbar^2}\right)^{1/8} e^{-(\sqrt{m\kappa}/2\hbar)x^2}$$

and the corresponding energy is given by $E_0 = \frac{1}{2}\hbar\omega_0$, where $\omega_0 = \sqrt{\kappa/m}$.

(a) Find the classical turning points for the particle with the total energy E_0 (values of x at which the classical particle would lose all of its speed and turn around).

$E_0 = \frac{1}{2}\kappa x_\pm^2$ at the two turning points

Then $x_\pm = \pm\sqrt{\frac{2E_0}{\kappa}} = \pm\sqrt{\frac{\hbar\omega_0}{\kappa}}$.

(b) What is the probability that the particle in the ground state is found in the $x \in [-1, 1]$ interval, in some arbitrary units? [Leave the answer in the integral form]

The prob. is given by

$$\int_{-1}^{1} dx\, |\psi(x)|^2 = \frac{(m\kappa)^{1/4}}{(\pi\hbar)^{1/2}} \int_{-1}^{1} dx\, e^{-\frac{\sqrt{m\kappa}}{\hbar}x^2}$$

[EXTRA CREDIT]

(c) Find the RMS of x, σ_x and the RMS of p, σ_p for the ground-state particle. Is the uncertainty principle satisfied?

Useful integrals:

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty} dx e^{-x^2/2\sigma^2} = 1, \quad (1)$$

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty} dx x e^{-x^2/2\sigma^2} = 0, \quad (2)$$

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty} dx x^2 e^{-x^2/2\sigma^2} = \sigma^2. \quad (3)$$

In our problem, $\frac{1}{2\sigma^2} = \frac{\sqrt{mk}}{\hbar} \Rightarrow$

$\Rightarrow \sigma^2 = \frac{\hbar}{2\sqrt{mk}}$

Then clearly $\langle x \rangle = 0$ (Eq. 2) and

$\langle x^2 \rangle = \sigma^2$ (Eq. 3). $\Rightarrow \sigma_x^2 = \sigma^2$

Recall that $\hat{p} = -i\hbar\frac{\partial}{\partial x} \Rightarrow$

$\Rightarrow \int_{-\infty}^{\infty} dx\, \Psi^*(x)\hat{p}\Psi(x) \sim \int_{-\infty}^{\infty} dx\, x\, e^{-x^2/2\sigma^2} = 0,$ as expected

Finally, $\int_{-\infty}^{\infty} dx\, \Psi^*(x) \underbrace{\left(-i\hbar\frac{\partial}{\partial x}\right)^2}_{\hat{p}^2} \Psi(x) =$

$= \frac{1}{\sqrt{2\pi\sigma^2}}(-\hbar^2)\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{4\sigma^2}} \frac{\partial}{\partial x}\left(-\frac{x}{2\sigma^2} e^{-\frac{x^2}{4\sigma^2}}\right) =$

$= \frac{-\hbar^2}{\sqrt{2\pi\sigma^2}}\left[\int_{-\infty}^{\infty} dx \left(-\frac{1}{2\sigma^2}\right) e^{-\frac{x^2}{2\sigma^2}} + \int_{-\infty}^{\infty} dx \frac{x^2}{4\sigma^4} e^{-\frac{x^2}{2\sigma^2}}\right] =$

$= \frac{\hbar^2}{2\sigma^2} - \frac{\hbar^2}{4\sigma^2} = \frac{\hbar^2}{4\sigma^2}.$

$\sigma_p^2 = \frac{\hbar^2}{4\sigma^2}$

Finally,

$\sigma_x^2\sigma_p^2 = \frac{\hbar^2}{4\sigma^2}\sigma^2 = \frac{\hbar^2}{4}$

$\sigma_x\sigma_p = \frac{\hbar}{2}$ minimum uncertainty!