Solution 8

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8.69 We know that for there to be a spin–orbit interaction ℓ must be nonzero, so that n must be at least 2. Let us use $\ell = 1$ and $r = 2^2 a_0$. $g_e \dfrac{\mu_0 e^2}{8\pi m_e^2 r^3} \mathbf{S}\cdot\mathbf{L} \sim 2\dfrac{\mu_0 e^2}{8\pi m_e^2 (4a_0)^3}\dfrac{\sqrt{3}}{2}\hbar\sqrt{2}\,\hbar$.

Since $\mu_0 = \dfrac{1}{\varepsilon_0 c^2}$, this energy becomes $\dfrac{\sqrt{6}}{128}\dfrac{\hbar^2 e^2}{(4\pi\varepsilon_0)c^2 m_e^2 a_0^3} = \dfrac{\sqrt{6}}{128}\left(\dfrac{e^2}{(4\pi\varepsilon_0)\hbar c}\right)^2 \dfrac{(4\pi\varepsilon_0)\hbar^4}{m_e^2 e^2 a_0^3}$.

Now, using $a_0 = \dfrac{(4\pi\varepsilon_0)\hbar^2}{m_e e^2}$ we obtain $\dfrac{\sqrt{6}}{128}\left(\dfrac{e^2}{(4\pi\varepsilon_0)\hbar c}\right)^2 \dfrac{(4\pi\varepsilon_0)\hbar^4}{m_e^2 e^2}\left(\dfrac{m_e e^2}{(4\pi\varepsilon_0)\hbar^2}\right)^3$

$= \dfrac{\sqrt{6}}{16}\left(\dfrac{e^2}{(4\pi\varepsilon_0)\hbar c}\right)^2 \dfrac{m_e e^4}{2(4\pi\varepsilon_0)^2\hbar^2}\dfrac{1}{2^2} = 0.15\,\alpha^2 E_2$.

8.72 j may be $\ell+\frac{1}{2}$ or $\ell-\frac{1}{2}$, giving respectively a $\mathbf{4f}_{5/2}$ and a $\mathbf{4f}_{7/2}$ state. As noted in Section 8.7, the state of higher j, where **L** and **S** are aligned, is of higher energy. **(b)** For a given j there are $2j+1$ values of m_j (i.e., from $-j$ to $+j$ in integral steps), which correspond to as many different orientation energies in the external field. For $j = 5/2$, $2j+1 = \mathbf{6}$, while for $j = 7/2$ it is **8**

8.75 From $J^2 = L^2 + S^2 + 2\,\mathbf{L}\cdot\mathbf{S}$ we have $\mathbf{L}\cdot\mathbf{S} = \frac{1}{2}(J^2 - L^2 - S^2)$. Substituting: $\mu_J\cdot\mathbf{J} = -\dfrac{e}{2m_e}\left(L^2 + 2S^2 + 3\dfrac{1}{2}(J^2 - L^2 - S^2)\right)$

$= -\dfrac{e}{2m_e}\left(\dfrac{3}{2}J^2 - \dfrac{1}{2}L^2 + \dfrac{1}{2}S^2\right) = -\dfrac{e}{2m_e}\left(\dfrac{3}{2}j(j+1)\hbar^2 - \dfrac{1}{2}\ell(\ell+1)\hbar^2 + \dfrac{1}{2}s(s+1)\hbar^2\right)$.

Thus, $\dfrac{|\mu_J\cdot\mathbf{J}|}{J} = \dfrac{\dfrac{e}{2m_e}\left(\dfrac{3}{2}j(j+1)\hbar^2 - \dfrac{1}{2}\ell(\ell+1)\hbar^2 + \dfrac{1}{2}s(s+1)\hbar^2\right)}{\sqrt{j(j+1)}\,\hbar} = \dfrac{e}{2m_e}\dfrac{3j(j+1) - \ell(\ell+1) + s(s+1)}{2\sqrt{j(j+1)}}\hbar$

9.26 There are six ways—(0,5), (1,4), (2,3), (3,2), (4,1) and (5,0)—and 6!/(5!1!) is indeed 6.

(b) There are 15 ways—(0,0,0,0,2), (0,0,0,2,0), (0,0,2,0,0), (0,2,0,0,0), (2,0,0,0,0), (0,0,0,1,1), (0,0,1,0,1), (0,1,0,0,1), (1,0,0,0,1), (0,0,1,1,0), (0,1,0,1,0), (1,0,0,1,0), (0,1,1,0,0), (1,0,1,0,0) and (1,1,0,0,0)—and 6!/(2!3!) is 15.

9.28 There are two ways to go here. Equation (9-12) gives the probability. The energy E_n is $n\hbar\omega_0$. Thus, $P(E_n) = \dfrac{e^{-n\hbar\omega_0/k_BT}}{\sum_{n=0}^{\infty} e^{-n\hbar\omega_0/k_BT}}$. The sum in the denominator can be simplified: $\dfrac{e^{-n\hbar\omega_0/k_BT}}{\sum_{n=0}^{\infty} x^n}$, where $x = e^{-\hbar\omega_0/k_BT}$. Using $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$, the probability becomes $\left(1 - e^{-\hbar\omega_0/k_BT}\right)e^{-n\hbar\omega_0/k_BT}$. For $n = 0$, i.e., for the ground state, this becomes $P(0) = \left(1 - e^{-\hbar\omega_0/k_BT}\right)$. We see that a larger T implies a smaller probability. At what T is it one–half? $\dfrac{1}{2} = \left(1 - e^{-\hbar\omega_0/k_BT}\right) \Rightarrow \ln\dfrac{1}{2} = -\dfrac{\hbar\omega_0}{k_BT}$ or $T = \dfrac{\hbar\omega_0}{\mathbf{k_B ln2}}$. The other route is to use (9-17). For $n = 0$, it becomes simply $P(0) = \dfrac{1}{1 + M/N}$. Rearranging (9-16) and inserting gives $P(0) = \left(1 - e^{-\hbar\omega_0/k_BT}\right)$, as above. As always, k_BT needs to be comparable to the jump between levels before the probability gets large.

9.36 There are $2n^2$ values of ℓ, m_ℓ and m_s for each n. The number of particles with energy E_n is the number of states times the Boltzmann occupation number: # with energy $E_n \propto 2n^2 e^{-E_n/k_BT}$ Thus: $\frac{\text{\# with energy } E_n}{\text{\# with energy } E_1} = \frac{2n^2 e^{-E_n/k_BT}}{2e^{-E_1/k_BT}}$

$= n^2 e^{-(E_n - E_1)/k_BT} = n^2 e^{-13.6\text{eV}\left(\frac{1}{n^2}-1\right)/k_BT}$

(b) As n becomes larger the $1/n^2$ approaches zero, so that the ratio becomes $n^2 e^{-13.6\text{eV}/k_BT}$. Given a **high enough n and/or T** this would exceed unity.

(c) At 6000K, k_BT = $(1.38\times10^{-23}\text{J/K})(6000\text{K})(6.25\times10^{18}\text{eV/J})$ = 0.5175eV. Thus $0.01 = n^2 e^{-13.6/0.5175} \Rightarrow$ n = **51,000**.

(d) The fifty–thousandth quantum level is essentially free. Taking into account ionized atoms would change the whole picture.

9.41 $\overline{v^2} = \int_0^\infty v^2 \left[\sqrt{\frac{2}{\pi}}\left(\frac{m}{k_BT}\right)^{3/2} v^2 e^{-\frac{1}{2}mv^2/k_BT}\right] dv = \sqrt{\frac{2}{\pi}}\left(\frac{m}{k_BT}\right)^{3/2} \int_0^\infty v^4 e^{-\frac{1}{2}mv^2/k_BT} dv = \sqrt{\frac{2}{\pi}}\left(\frac{m}{k_BT}\right)^{3/2} \int_0^\infty z^4 e^{-az^2} dz$ where $a \equiv \frac{m}{2k_BT}$.

The Gaussian integral is $-\frac{d^2}{da^2}\int_0^\infty e^{-az^2}dz = -\frac{d^2}{da^2}\left(\frac{1}{2}\sqrt{\frac{\pi}{a}}\right) = \frac{1}{2a^2} = \frac{3}{8}\sqrt{\frac{\pi}{a^5}}$. Thus $\overline{v^2} = \sqrt{\frac{2}{\pi}}\left(\frac{m}{k_BT}\right)^{3/2}\frac{3}{8}\sqrt{\pi}\left(\frac{2k_BT}{m}\right)^{5/2}$

$= \frac{3k_BT}{m}$ and $v_{rms} = \sqrt{\overline{v^2}} = \sqrt{\frac{3k_BT}{m}}$