Solution 5

Physics 313

5.83 $\bar{x} = \int_{\text{all scape}} x\psi^2(x)\,dx = \int_0^\infty x\left(2\sqrt{a^3}xe^{-ax}\right)^2 dx = 4a^3\int_0^\infty x^3e^{-2ax}dx = 4a^3\frac{3!}{(2a)^4} = \frac{1.5}{a}$

$$\overline{x^2} = \int_{\text{all scape}} x^2\psi^2(x)\,dx = \int_0^\infty x^2\left(2\sqrt{a^3}xe^{-ax}\right)^2 dx = 4a^3\int_0^\infty x^4e^{-2ax}dx = 4a^3\frac{4!}{(2a)^5} = \frac{3}{a^2}$$

$$\Delta x = \sqrt{\overline{x^2}-\bar{x}^2} = \sqrt{\frac{3}{a^2}-\left(\frac{1.5}{a}\right)^2} = \frac{\sqrt{0.75}}{a} = \frac{\mathbf{0.866}}{\mathbf{a}}$$

5.85 $\bar{p} = \int_0^\infty \left(2\sqrt{a^3}xe^{-ax}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\left(2\sqrt{a^3}xe^{-ax}\right)dx = 4a^3(-i\hbar)\int_0^\infty\left(xe^{-ax}\right)\left((1-ax)e^{-ax}\right)dx$

$$= 4a^3(-i\hbar)\left(\int_0^\infty xe^{-2ax}dx - a\int_0^\infty x^2e^{-2ax}dx\right) = 4a^3(-i\hbar)\left(\frac{1!}{(2a)^2}-a\frac{2!}{(2a)^3}\right) = \mathbf{0}.$$

$$\overline{p^2} = \int_0^\infty \left(2\sqrt{a^3}xe^{-ax}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)^2\left(2\sqrt{a^3}xe^{-ax}\right)dx = 4a^3(-\hbar^2)\int_0^\infty (a^2x^2-2ax)e^{-2ax}dx$$

$$= -4a^2\hbar^2\left(a^2\frac{2!}{(2a)^3}-2a\frac{1!}{(2a)^2}\right) = a^2\hbar^2$$

$$\Delta p = \sqrt{\overline{p^2}-\bar{p}^2} = \mathbf{a\hbar}$$

5.86 $\Delta x\,\Delta p = \frac{0.866}{a}a\hbar = \mathbf{0.866\hbar}$. The product is $\geq\frac{1}{2}\hbar$, as it must be. Since the wave function is not a Gaussian, it should indeed be greater than the minimum product of $\frac{1}{2}\hbar$.

6.13 $A'e^{ikx}+B'e^{-ikx} = A'(\cos kx + i\sin kx) + B'(\cos kx - i\sin kx) = (A'+B')\cos kx + i(A'-B')\sin kx$

$$= (\tfrac{1}{2}[B-iA]+\tfrac{1}{2}[B+iA])\cos kx + i(\tfrac{1}{2}[B-iA]-\tfrac{1}{2}[B+iA])\sin kx = B\cos kx + A\sin kx$$

6.17 To left of step (x < 0), $\psi = \psi_{\text{inc}} + \psi_{\text{refl}} = 1\,e^{ikx} + B\,e^{-ikx}$. To right (x > 0), $\psi = C\,e^{-\alpha x}$.

ψ must be continuous at x = 0: $1\,e^0 + B\,e^0 = C\,e^0 \Rightarrow 1 + B = C$.

$\frac{d\psi}{dx}$ must be continuous at x = 0: $ik\,1\,e^0 - ik\,B\,e^0 = -\alpha C\,e^0 \Rightarrow ik\,(1-B) = -\alpha C$.

Substituting for C in second, using first: $ik\,(1-B) = -\alpha(1+B) \Rightarrow B = \frac{ik+\alpha}{ik-\alpha} = \frac{i\sqrt{2mE}/\hbar+\sqrt{2m(\frac{5}{4}E-E)}/\hbar}{i\sqrt{2mE}/\hbar-\sqrt{2m(\frac{5}{4}E-E)}/\hbar}$.

Dividing everywhere by $\sqrt{2mE}/\hbar$, $B = \frac{i+\sqrt{\frac{5}{4}-1}}{i-\sqrt{\frac{5}{4}-1}} = \frac{i+(1/2)}{i-(1/2)} = \frac{3}{5}-i\frac{4}{5}$.

Now plugging back in: $C = 1+B = \frac{8}{5}-i\frac{4}{5}$. $\psi_{\text{refl}} = \left(\frac{\mathbf{3}}{\mathbf{5}}-\mathbf{i}\frac{\mathbf{4}}{\mathbf{5}}\right)\mathbf{e}^{-\mathbf{i}kx}$, where $k = \frac{\sqrt{2mE}}{\hbar}$.

(a) $\psi_{x>0} = \left(\frac{\mathbf{8}}{\mathbf{5}}-\mathbf{i}\frac{\mathbf{4}}{\mathbf{5}}\right)\mathbf{e}^{-\alpha x}$, where $\alpha = \frac{\sqrt{2m(U_0-E)}}{\hbar}$

(b) $B^*B = \left(\frac{3}{5}+i\frac{4}{5}\right)\left(\frac{3}{5}-i\frac{4}{5}\right) = 1$

6.25 This is not tunneling; the kinetic energy is never negative and the wave function between 0 and L is thus of the form $e^{ik'x}$ not $e^{-\alpha x}$. Therefore, we need only replace U_0 by $-U_0$ in the potential barrier reflection equation (6-13).

$$R = \frac{\sin^2\left(\frac{\sqrt{2m(E+U_0)}}{\hbar}L\right)}{\sin^2\left(\frac{\sqrt{2m(E+U_0)}}{\hbar}L\right)+4\frac{E}{U_0}\left(\frac{E}{U_0}+1\right)}$$

6.26 As $U_0 \to \infty$, the term $4\frac{E}{U_0}\left(\frac{E}{U_0}+1\right)$ in the denominator approaches zero, leaving simply sine squared over sine squared.

(b) As $L \to 0$, the sine squared factor in the numerator approaches zero, while the denominator approaches the presumably finite $4\frac{E}{U_0}\left(\frac{E}{U_0}+1\right)$

(c) If U_0L is constant, then so is $\sqrt{U_0L}$, so that the product $\sqrt{U_0L}\sqrt{L}$ in the argument of the sine factors approaches zero. The approximation $\sin x \cong x$ is then appropriate.

$$R \cong \frac{\frac{2m(E+U_0)}{\hbar^2}L^2}{\frac{2m(E+U_0)}{\hbar^2}L^2 + 4\frac{E}{U_0}\left(\frac{E}{U_0}+1\right)}.$$

Replacing $E + U_0$ by U_0 and $\frac{E}{U_0}+1$ by 1 gives $R \cong \frac{\frac{2mU_0}{\hbar^2}L^2}{\frac{2mU_0}{\hbar^2}L^2 + 4\frac{E}{U_0}} = \frac{1}{1+\frac{2\hbar^2 E}{m(U_0L)^2}}$