Solution 4

Physics 313

5.24 $E_4 - E_1 = \frac{\pi^2\hbar^2}{2mL^2}(4^2 - 1^2) = \frac{\pi^2(1.055\times10^{-34}\,\text{J·s})^2}{2(9.11\times10^{-31}\,\text{kg})(5\times10^{-9}\,\text{m})^2}(15) = 3.6\times10^{-20}\,\text{J} = 0.226\,\text{eV}$

$E = h\frac{c}{\lambda} \rightarrow 3.6\times10^{-20}\,\text{J} = 6.63\times10^{-34}\,\text{J·s}\,\frac{3\times10^8\,\text{m/s}}{\lambda} \Rightarrow \lambda = \mathbf{5.5\times10^{-6}\,m}$ (Infrared.)

5.25 Since the energy levels get further apart as n increases, the lowest energy transition will be from the $n = 2$ level to the $n = 1$. The photon's energy is $hf = h\frac{c}{\lambda}$. This equals the energy difference between the two levels, $E_2 - E_1 = \frac{\pi^2\hbar^2}{2mL^2}(2^2 - 1^2)$. Thus, $(6.63\times10^{-34}\,\text{J·s})\frac{3\times10^8\,\text{m}}{450\times10^{-9}\,\text{m}} = \frac{\pi^2(1.055\times10^{-34}\,\text{J·s})^2}{2(9.11\times10^{-31}\,\text{kg})L^2}\times 3 \Rightarrow L = 6.4\times10^{-10}\,\text{m} = \mathbf{0.64nm}$.

5.28 $\psi_2(x) = \sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L}$. Prob $= \int|\psi_2(x)|^2\,dx = \frac{2}{L}\int_{\frac{1}{3}L}^{\frac{2}{3}L}\sin^2\frac{2\pi x}{L}\,dx = \frac{2}{L}\left(\frac{x}{2} - L\frac{\sin\frac{4\pi x}{L}}{8\pi}\right)\Bigg|_{\frac{1}{3}L}^{\frac{2}{3}L}$

$$= \frac{2}{L}\left(\frac{L}{6} - L\frac{\sin\frac{8\pi}{3} - \sin\frac{4\pi}{3}}{8\pi}\right) = \frac{1}{3} - 0.138 = \mathbf{0.196.}$$

Classically, it should be one third. This is lower because the region is centered on a node.

5.30 Wave function outside must be zero. Inside: $\psi(x) = A\sin kx + B\cos kx$. Must be 0 both at $x = +\frac{1}{2}a$ and $-\frac{1}{2}a$. $A\sin(k(-\frac{1}{2}a)) + B\cos(k(-\frac{1}{2}a)) = 0$ and $A\sin(k(+\frac{1}{2}a)) + B\cos(k(+\frac{1}{2}a)) = 0$. Or, $B\cos(\frac{1}{2}a) \pm A\sin(\frac{1}{2}a) = 0$. Both $A\sin(\frac{1}{2}a)$ and $B\cos(\frac{1}{2}a)$ have to be zero! We cannot have both A and B zero at once, or we would have no wave! And sine and cosine are never zero at same place, so we cannot have both A and B nonzero. The only possibilities are: (1) cosine is zero when A is zero, and (2) sine is zero when B is zero.

(1) $\cos(\frac{1}{2}a) = 0 \Rightarrow \frac{1}{2}ka = n\frac{\pi}{2}$ (n odd) $\Rightarrow k = \frac{n\pi}{a}$ but, $k = \frac{\sqrt{2mE}}{\hbar} \Rightarrow \frac{n^2\pi^2}{a^2} = \frac{2mE}{\hbar^2} \Rightarrow E = \frac{n^2\pi^2\hbar^2}{2mL^2}$

(2) $\sin(\frac{1}{2}a) = 0 \Rightarrow \frac{1}{2}ka = n\frac{\pi}{2}$ (n even). This gives again $E = \frac{n^2\pi^2\hbar^2}{2ma^2}$ and just fills in the even n.

Normalize: $\int_{-\frac{1}{2}a}^{+\frac{1}{2}a} A^2\sin^2\frac{n\pi x}{a}\,dx = A^2\frac{a}{2} = 1 \Rightarrow A = \sqrt{\frac{2}{a}}$ and $\int_{-\frac{1}{2}a}^{+\frac{1}{2}a} B^2\cos^2\frac{n\pi x}{a}\,dx = B^2\frac{a}{2} = 1 \Rightarrow B = \sqrt{\frac{2}{a}}$

$\boldsymbol{\psi(x) = \sqrt{\frac{2}{a}}\cos\frac{n\pi x}{a}\,(n\ \text{odd})}$, $\boldsymbol{\psi(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}\,(n\ \text{even})}$, $\mathbf{E = \frac{n^2\pi^2\hbar^2}{2ma^2}}$. When plotted, these look like infinite well wave functions, because it *is* an infinite well; it's just moved sideways $\frac{1}{2}L$.

5.34 $\delta = \frac{\hbar}{\sqrt{2m(U_0 - E)}} = \frac{1.055\times10^{-34}\,\text{J·s}}{\sqrt{2(9.11\times10^{-31}\,\text{kg})(200\text{eV} - 50\text{eV})1.6\times10^{-19}\,\text{J/eV}}} = \mathbf{1.6\times10^{-11}\,m}$

5.51 $\Delta E = \hbar\omega_0 = \hbar\sqrt{\frac{\kappa}{m}} = 1.055\times10^{-34}\sqrt{\frac{2.3\times10^3\,\text{N/m}}{\frac{1}{2}(14\times1.66\times10^{-27}\,\text{kg})}} = 4.69\times10^{-20}\,\text{J}$. Equating to $\frac{3}{2}k_BT$, we have $4.69\times10^{-20}\,\text{J} = \frac{3}{2}(1.38\times10^{-23}\,\text{J·s})T \Rightarrow T \cong 2{,}300\text{K}$. T would have to be thousands of Kelvin to excite non–ground oscillator levels.

5.55 $\bar{x} = \Sigma x \,\text{Prob}(x) = \Sigma x\left(\frac{\text{prob}}{\text{dx}}dx\right) \rightarrow \int_0^L x\frac{1}{L}dx = \frac{1}{2}L$ and $\overline{x^2} = \Sigma x^2\left(\frac{\text{prob}}{\text{dx}}dx\right) \rightarrow \int_0^L x^2\frac{1}{L}dx = \frac{1}{3}L^2$.

Thus, $\Delta x = \sqrt{\overline{x^2} - \bar{x}^2} = \sqrt{\frac{1}{3}L^2 - \frac{1}{4}L^2} = \frac{1}{\sqrt{12}}L$

5.56 $\bar{x} = \int\limits_{\text{all space}} \psi^* x \psi dx = \int_0^L \left(\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\right)x\left(\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L x\sin^2\frac{n\pi x}{L}dx$

$$= \frac{2}{L}\int_0^L x\frac{1-\cos(2n\pi x/L)}{2}dx = \frac{2}{L}\left(\frac{x^2}{4} - x\frac{\sin(2n\pi x/L)}{2(2n\pi/L)} - \frac{\cos(2n\pi x/L)}{2(2n\pi/L)^2}\right)\Bigg|_0^L = \frac{L}{2}$$

(Second and third terms, obtained via integration by parts, are each separately zero.)

$$\overline{x^2} = \int\psi^* x^2\psi dx = \frac{2}{L}\int_0^L x^2\sin^2\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L x^2\frac{1-\cos(2n\pi x/L)}{2}dx$$

$$= \frac{2}{L}\left(\frac{x^3}{6} - x^2\frac{\sin(2n\pi x/L)}{2(2n\pi/L)} - 2x\frac{\cos(2n\pi x/L)}{2(2n\pi/L)^2} + 2\frac{\sin(2n\pi x/L)}{2(2n\pi/L)^3}\right)\Bigg|_0^L = \frac{2}{L}\left(\frac{L^3}{6} - 0 - 2L\frac{\cos(2n\pi)}{2(2n\pi/L)^2} + 0\right)$$

$$= \frac{L^2}{3} - \frac{L^2}{2n^2\pi^2}$$

$$\Delta x = \sqrt{\overline{x^2} - \bar{x}^2} = \sqrt{\frac{L^2}{3} - \frac{L^2}{2n^2\pi^2} - \frac{L^2}{4}} = L\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}.$$

As $n \rightarrow \infty$, this approaches the classical uncertainty calculated in Exercise 55.