Physics 313
Final Exam

December 18, 2015

The exam is 3 hours in length.

There is a total of 9 problems (8 mandatory + 1 extra credit). Each problem is worth 10 points regardless of its length or the number of parts.

You may refer ONLY to two double-sided or four single-sided sheets of paper with notes (US Letter size) that you brought with you. You may also use a calculator.

Do not forget to write your name on the first page!

Good luck!

Problem 1

Consider a relativistic macroscopic object of mass $m = 10^{-3}$ kg moving with the velocity $u_1 = 0.8c$.

(a) What is the total energy of this object?

$E = \gamma_{u_1} mc^2$, where $\gamma_{u_1} = \dfrac{1}{\sqrt{1 - u_1^2/c^2}}$.

In this case, $\gamma_{u_1} \simeq 1.67$ & therefore

$E = 1.67 \times 10^{-3}\,\mathrm{kg} \times (3 \times 10^8 \,\frac{\mathrm{m}}{\mathrm{s}})^2 =$

$\simeq 1.5 \times 10^{14}$ J.

(b) What are its internal and kinetic energies? What is the speed of the object, u_2, at which the two energies would be equal?

The internal energy is

$E_{rest} = mc^2 = 9 \times 10^{13}$ J.

The kinetic energy is then given by

$KE = E - E_{rest} = 6 \times 10^{13}$ J.

For $KE = E_{rest}$, we obtain

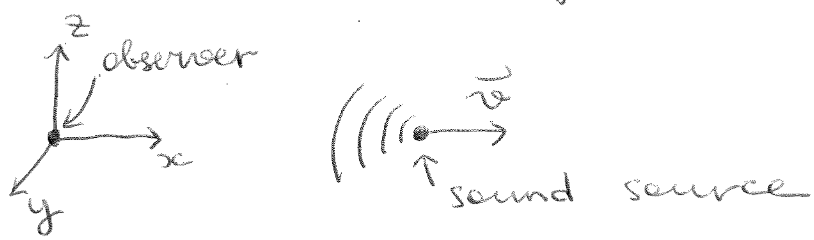
$(\gamma_{u_2} - 1) mc^2 = mc^2 \Rightarrow \gamma_{u_2} = 2 \Rightarrow u_2 = \frac{\sqrt{3}}{2} c \approx$

$\approx 0.87c$.

Problem 2

An observer stands on the ground. A sound source is moving *away* from the observer with the speed $v \ll c$; in the frame of the source, the sound frequency is f. Calculate the sound frequency in the observer's frame. How does the answer change if the source is moving *toward* the observer with the same speed? Which frequency is higher?

c – speed of sound



The time between 2 pulses in the frame of the sound source: Δt.

Then the time between the same 2 pulses recorded by the observer:

$$\Delta t' = \Delta t + \underbrace{\frac{v}{c}\Delta t}_{\text{extra time due to source motion}} = \left(1 + \frac{v}{c}\right)\Delta t .$$

For continuous signals,

$$f \sim 1/\Delta t \quad \& \quad f' \sim 1/\Delta t' .$$

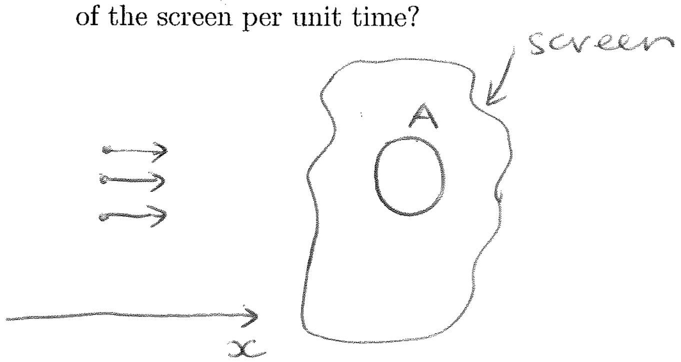
Then $$f' = f\,\frac{1}{1+\frac{v}{c}} \underset{v \ll c}{\approx} f\left(1 - \frac{v}{c}\right)$$

If the source is moving towards the observer, $v \to -v \Rightarrow f' = f\left(1 + \frac{v}{c}\right)$

Clearly, $f'_{\text{toward}} > f'_{\text{away}}$.

Problem 3

Consider a flux of particles moving left to right along the x-axis: each particle has the same velocity v and charge q, and the particle density (the number of particles per unit volume) is n. Calculate the current density j (current per unit area) in such a system. Hint: Imagine putting up a screen perpendicular to the x-axis. How many particles impinge on the area A of the screen per unit time?



Clearly, all the particles within volume vA will reach the area A of the screen in unit time.

There are nvA such particles, and they carry the total charge of $qnvA$. Then the current density (charge per unit area per unit time) is simply

$$j = qnv \quad (\text{or } \vec{j} = qn\vec{v})$$

Problem 4

Recall that the solution of the radial Schrodinger equation for the hydrogen atom is given by

$$R_{n,l}(r) = \frac{2}{(2a_0)^{3/2}}\left(1 - \frac{r}{2a_0}\right)e^{-r/2a_0}$$

for $n = 2$, $l = 0$ (a_0 is the Bohr radius). What is the radial probability (probability per unit radial distance) of an electron in this state? Find the average radial distance r_{ave} of the electron.

Useful integral:

$$\int_0^\infty dx x^m e^{-bx} = \frac{m!}{b^{m+1}}$$

The radial probability is given by

$$P(r) = r^2 R^2_{2,0}(r) = \frac{1}{2a_0^3} r^2 \left(1 - \frac{r}{2a_0}\right)^2 e^{-r/a_0}.$$

Then $r_{ave} = \int_0^\infty dr\, r\, P(r) = \frac{1}{2a_0^3}\left[\int_0^\infty dr\, r^3 e^{-r/a_0} - \right.$

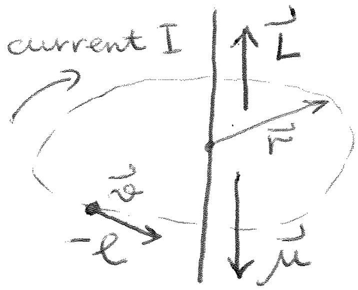
$\left. - \frac{1}{a_0}\int_0^\infty dr\, r^4 e^{-r/a_0} + \frac{1}{4a_0^2}\int_0^\infty dr\, r^5 e^{-r/a_0}\right] =$

$= \frac{1}{2a_0^3}\left[a_0^4\, 3! - \frac{a_0^5}{a_0} 4! + \frac{a_0^6}{4a_0^2} 5!\right] =$

$= 6a_0.$

Problem 5

Consider a classical electron of charge $-e$ orbiting counter-clockwise with a constant speed v in a circle of radius r. Find the electron's magnetic dipole moment $\vec{\mu}$ in terms of its angular momentum $\vec{L}$.



Negatively charged $\bar{e}$ orbiting counter-clockwise produces clockwise current

$$I = \frac{e}{\underset{\text{period}}{T}} = \frac{e}{2\pi r/v}.$$

The magnetic dipole moment is

$$\mu = I\underset{\text{area}}{A} = \frac{e}{2\pi r/v}\pi r^2 = \frac{e}{2}rv =$$

$$= \frac{e}{2\underset{\bar{e}\text{ mass}}{m_e}} m_e r v = \frac{e}{2m_e}L.$$

$\vec{L} = \vec{r}\times\vec{p}$ points up, and $\vec{\mu}$ points down because $\vec{I}$ is clockwise:

$$\vec{\mu} = -\frac{e}{2m_e}\vec{L}$$

Problem 6

Suppose that two indistinguishable quantum particles are described by *spatial* wavefunctions $\psi_n(\vec{x}_1)$ and $\psi_{n'}(\vec{x}_2)$, respectively (n and n' are quantum numbers fully characterizing the spatial state). Assume that the particles do not interact with one another.

(a) Given that the particles have spin $s = 0$, write down the total *spatial* wavefunction of the two-particle system.

Recall that $s=0$ particles are bosons, and thus the total wavefunction is symmetric wrt particle labels. But the spin part is explicitly symmetric, so the spatial part must be symmetric too:

$$\Psi_S(\vec{x}_1, \vec{x}_2) = \Psi_n(\vec{x}_1)\Psi_{n'}(\vec{x}_2) + \Psi_n(\vec{x}_2)\Psi_{n'}(\vec{x}_1)$$

(b) Now assume that the particles have spin $s = 1/2$. Write down the *spatial* wavefunction for the case of *parallel* spins in the two-particle system. Is this wavefunction symmetric, antisymmetric, or neither with respect to exchanging particle labels? Discuss the situation in which $n = n'$.

$s=\frac{1}{2}$ particles are fermions, and therefore their total wavefunction must be antisymmetric. Since the spin part is symmetric for parallel spins $(\uparrow\uparrow)$, the spatial part must be antisymmetric:

$$\Psi_A(\vec{x}_1, \vec{x}_2) = \Psi_n(\vec{x}_1)\Psi_{n'}(\vec{x}_2) - \Psi_n(\vec{x}_2)\Psi_{n'}(\vec{x}_1)$$

If $n=n'$, this wavefunction is identically 0 $\Rightarrow$ Pauli exclusion principle! (fermions cannot occupy the same state if they are indistinguishable)

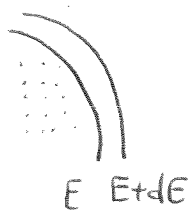
Problem 7

Consider Fermi-Dirac quantum gas of spin-1/2 particles of mass m at $T = 0$. Assume that the particles are in the infinite 3D well (i.e., they are contained inside a cube of the volume $V = L^3$).

(a) Compute the density of states, $D(E)$, for this system.

without spin, $E = \frac{\pi^2\hbar^2}{2mL^2}\underbrace{n^2}_{n_x^2+n_y^2+n_z^2} \Rightarrow n = \sqrt{\frac{2mL^2E}{\pi^2\hbar^2}}$.

Fermi sphere:



$$dE = \underbrace{\frac{1}{8}}_{n_{x,y,z}>0}\ \underbrace{4\pi n^2 dn}_{\text{volume of a shell of thickness } dn}$$

$$D(E) = \frac{\pi n^2}{2}\frac{dn}{dE} = \frac{\pi}{2}\left(\frac{2mL^2E}{\pi^2\hbar^2}\right)\sqrt{\frac{2mL^2}{\pi^2\hbar^2}}\,\frac{1}{2\sqrt{E}} = \frac{m^{3/2}L^3}{\sqrt{2}\,\pi^2\hbar^3}\sqrt{E}.$$

(b) Using $D(E)$, write down the expression for the total number of particles N at $T = 0$. Express the Fermi energy E_F in terms of the particle density, $n = N/V$.

With spin, multiply by $(2S+1) = 2$:

$$D(E) = \frac{\sqrt{2}\, m^{3/2}L^3}{\pi^2\hbar^3}\sqrt{E}.$$

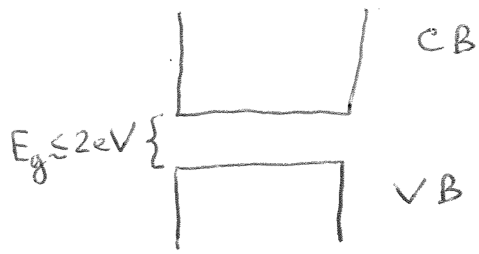
Now, at $T=0$ $\quad N = \int_0^\infty dE\, D(E)\, N_{FD}(E)$ becomes

$$N = \int_0^{E_F} dE\, D(E) = \frac{\sqrt{2}\, m^{3/2}L^3}{\pi^2\hbar^3}\,\frac{2}{3}E_F^{3/2}, \text{ or}$$

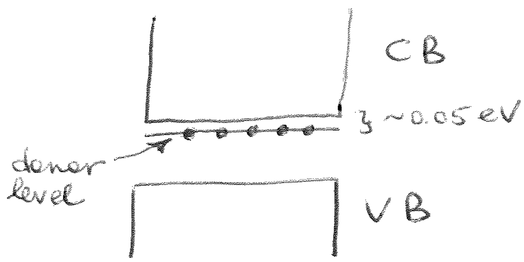
$$E_F = \frac{\pi^2\hbar^2}{2m}\left[\frac{3}{\pi}\frac{N}{\underset{L^3}{\underset{\shortparallel}{V}}}\right]^{2/3} = \left(\frac{3}{\pi}\right)^{2/3}\frac{\pi^2\hbar^2}{2m}n^{2/3}.$$

Problem 8

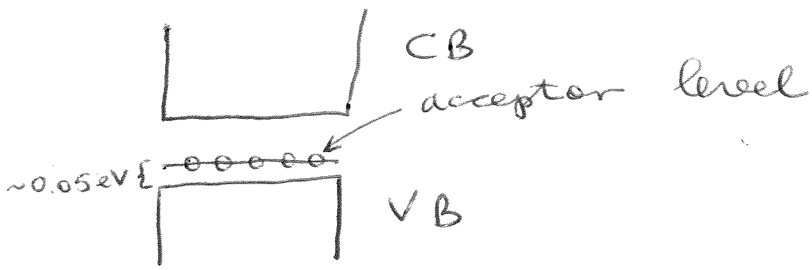
Sketch the band structure for an intrinsic semiconductor, n-type extrinsic semiconductor, and p-type extrinsic semiconductor (clearly label all three cases!). What are the majority carriers in the n-type and p-type semiconductors, and why?



intrinsic SC



n-type extrinsic SC



p-type extrinsic SC

at finite T, $\bar{e}$'s from the donor level get excited into the conduction band in n-type SC $\Rightarrow$ the majority carriers are $\bar{e}$'s.

Similarly, $\bar{e}$'s from the valence band get excited into the acceptor level at finite T in p-type SC $\Rightarrow$ the majority carriers are holes in the valence band.

Problem 9 [EXTRA CREDIT]

Hydrogen-like atoms are single-electron atoms with a nuclear charge of $+Ze$.

(a) Write down the energy levels of a hydrogen-like atom in terms of the energy levels of the hydrogen atom. What is the dependence of energy on Z?

Recall that in the hydrogen atom,

$$E_n = -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2}\frac{1}{n^2} = -\frac{13.6\text{ eV}}{n^2}, \quad n = 1, 2, 3, \ldots$$

If $e^2 \rightarrow Ze^2$, $E_n = -\frac{13.6\text{ eV}}{n^2}Z^2 \sim Z^2$.

Energies are deeper by a factor of Z^2.

(b) Recall that for $l = n-1$ states of the hydrogen atom, the radial wavefunction is given by

$$R_{n,n-1}(r) \sim r^{n-1}e^{-r/na_0},$$

where a_0 is the Bohr radius.

Compute the radius, r_n, at which the radial probability is maximum for the hydrogen atom. Using this result, find the radius at which the radial probability is maximum for the hydrogen-like atom. What is its dependence on Z?

The radial probability $P(r) = r^2 R^2_{n,n-1}(r) \sim$

$$\sim r^{2n}e^{-2r/na_0}$$

$$\left.\frac{dP(r)}{dr}\right|_{r_{max}} = 0 \Rightarrow 2n\, r_{max}^{2n-1} - \frac{2}{na_0} r_{max}^{2n} = 0, \text{ or}$$

$$r_{max} = n^2 a_0.$$

Here, $a_0 = \frac{(4\pi\epsilon_0)\hbar^2}{me^2}$ is the Bohr radius.

Clearly, $e^2 \rightarrow Ze^2$ produces

$$r_{max} = \frac{n^2 a_0}{Z} \sim \frac{1}{Z}.$$

The radii are smaller by a factor of Z.