Physics 313
Final Exam

December 18, 2015

The exam is 3 hours in length.

There is a total of 9 problems (8 mandatory + 1 extra credit). Each problem is worth 10 points regardless of its length or the number of parts.

You may refer ONLY to two double-sided or four single-sided sheets of paper with notes (US Letter size) that you brought with you. You may also use a calculator.

Do not forget to write your name on the first page!

Good luck!
Problem 1

Consider a relativistic macroscopic object of mass \( m = 10^{-3} \) kg moving with the velocity \( \mathbf{u}_4 = 0.8c \).

(a) What is the total energy of this object?

\[
E = \gamma u_1 mc^2, \quad \text{where} \quad \gamma u_1 = \frac{1}{\sqrt{1-\frac{u_1^2}{c^2}}}
\]

In this case, \( \gamma u_1 \approx 1.67 \) & therefore

\[
E = 1.67 \times 10^{-3} \text{ kg} \times \left( 3 \times 10^8 \frac{m}{s} \right)^2 =
\]

\[
= 1.5 \times 10^{14} \text{ J}.
\]

(b) What are its internal and kinetic energies? What is the speed of the object, \( \mathbf{u}_2 \), at which the two energies would be equal?

The internal energy is

\[
E_{\text{rest}} = mc^2 = 9 \times 10^{13} \text{ J}.
\]

The kinetic energy is then given by

\[
KE = E - E_{\text{rest}} = 6 \times 10^{13} \text{ J}.
\]

For \( KE = E_{\text{rest}} \), we obtain

\[
(\gamma u_2 - 1) mc^2 = mc^2 \Rightarrow \gamma u_2 = 2 \Rightarrow u_2 = \frac{\sqrt{3}}{2} c \approx
\]

\[
\approx 0.87c.
\]
Problem 2

An observer stands on the ground. A sound source is moving away from the observer with the speed \( v \ll c \); in the frame of the source, the sound frequency is \( f \). Calculate the sound frequency in the observer's frame. How does the answer change if the source is moving toward the observer with the same speed? Which frequency is higher?

\[ c - \text{speed of sound} \]

The time between 2 pulses in the frame of the sound source: \( \Delta t \).

Then the time between 2 pulses recorded by the observer:

\[ \Delta t' = \Delta t + \frac{v}{c} \Delta t = (1 + \frac{v}{c}) \Delta t. \]

\( \frac{\text{extra time due to source motion}}{} \)

For continuous signals,

\[ f \sim \frac{1}{\Delta t} \quad \text{and} \quad f' \sim \frac{1}{\Delta t'}. \]

\[ f' = f \frac{1}{1 + \frac{v}{c}} \approx f (1 - \frac{v}{c}) \]

\( \uparrow \quad \text{v \ll c} \)

If the source is moving towards the observer, \( v \rightarrow -v \) \( \Rightarrow f' = f (1 + \frac{v}{c}) \)

\[ \text{Clearly, } f'_{\text{toward}} > f'_{\text{away}}. \]
Problem 3

Consider a flux of particles moving left to right along the x-axis: each particle has the same velocity \( v \) and charge \( q \), and the particle density (the number of particles per unit volume) is \( n \). Calculate the current density \( j \) (current per unit area) in such a system. Hint: Imagine putting up a screen perpendicular to the x-axis. How many particles impinge on the area \( A \) of the screen per unit time?

![Diagram]

Clearly, all the particles within volume \( vA \) will reach the area \( A \) of the screen in unit time.

There are \( nVA \) such particles, and they carry the total charge \( qnVA \). Then the current density (charge per unit area per unit time) is simply

\[ j = qn \nu \quad \text{or} \quad \vec{j} = qn \vec{v} \]

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Problem 4

Recall that the solution of the radial Schrödinger equation for the hydrogen atom is given by

\[ R_{n,l}(r) = \frac{2}{(2a_0)^{3/2}} \left( 1 - \frac{r}{2a_0} \right) e^{-r/2a_0} \]

for \( n = 2, \ l = 0 \) (\( a_0 \) is the Bohr radius). What is the radial probability (probability per unit radial distance) of an electron in this state? Find the average radial distance \( r_{\text{ave}} \) of the electron.

Useful integral:

\[ \int_0^\infty x^m e^{-bx} \, dx = \frac{m!}{b^{m+1}} \]

The radial probability is given by

\[ P(r) = r^2 R^2_{2,0}(r) = \frac{1}{2a_0^3} \, r^2 \left( 1 - \frac{r}{2a_0} \right)^2 e^{-r/2a_0} \]

Then

\[ r_{\text{ave}} = \int_0^\infty dr \, r \, P(r) = \frac{1}{2a_0^3} \left[ \int_0^\infty dr \, r^3 e^{-r/2a_0} - \frac{1}{a_0} \int_0^\infty dr \, r^4 e^{-r/2a_0} + \frac{1}{4a_0^2} \int_0^\infty dr \, r^5 e^{-r/2a_0} \right] = \]

\[ = \frac{1}{2a_0^3} \left[ a_0^4 \, 3! - \frac{a_0^5}{a_0} \, 4! + \frac{a_0^6}{4a_0^2} \, 5! \right] = 6a_0. \]
Problem 5

Consider a classical electron of charge \(-e\) orbiting counter-clockwise with a constant speed \(v\) in a circle of radius \(r\). Find the electron’s magnetic dipole moment \(\vec{\mu}\) in terms of its angular momentum \(\vec{L}\).

\[
\vec{I} = \frac{\vec{L}}{2\pi r^2/T} = \frac{e}{2\pi r^2/T}
\]

The magnetic dipole moment is
\[
\vec{\mu} = \vec{I} \times \vec{A} = \frac{e}{2\pi r^2/T} \vec{A}
\]

\[
= \frac{e}{2m_e} \vec{v} \times \vec{r} = \frac{e}{2m_e} \vec{L}
\]

\(\vec{L} = \vec{r} \times \vec{p}\) points up, and \(\vec{\mu}\) points down because \(\vec{I}\) is clockwise:

\[
\vec{\mu} = -\frac{e}{2m_e} \vec{L}
\]
Problem 6

Suppose that two indistinguishable quantum particles are described by spatial wavefunctions $\psi_n(\vec{x}_1)$ and $\psi_{n'}(\vec{x}_2)$, respectively ($n$ and $n'$ are quantum numbers fully characterizing the spatial state). Assume that the particles do not interact with one another.

(a) Given that the particles have spin $s = 0$, write down the total spatial wavefunction of the two-particle system.

Recall that $s=0$ particles are bosons, and thus the total wavefunction is symmetric wrt particle labels.
But the spin part is explicitly symmetric, so the spatial part must be symmetric too:

$$\psi_s(\vec{x}_1, \vec{x}_2) = \psi_n(\vec{x}_1)\psi_{n'}(\vec{x}_2) + \psi_{n'}(\vec{x}_2)\psi_n(\vec{x}_1)$$

(b) Now assume that the particles have spin $s = 1/2$. Write down the spatial wavefunction for the case of parallel spins in the two-particle system. Is this wavefunction symmetric, antisymmetric, or neither with respect to exchanging particle labels? Discuss the situation in which $n = n'$.

$s = \frac{1}{2}$ particles are fermions, and therefore their total wavefunction must be antisymmetric. Since the spin part is symmetric for parallel spins (parallel), the spatial part must be antisymmetric:

$$\psi_A(\vec{x}_1, \vec{x}_2) = \psi_n(\vec{x}_1)\psi_{n'}(\vec{x}_2) - \psi_{n'}(\vec{x}_2)\psi_n(\vec{x}_1)$$

If $n = n'$, this wavefunction is identically 0, implying Pauli exclusion principle! (fermions cannot occupy the same state if they are indistinguishable)
Problem 7

Consider Fermi-Dirac quantum gas of spin-1/2 particles of mass \( m \) at \( T = 0 \). Assume that the particles are in the infinite 3D well (i.e., they are contained inside a cube of the volume \( V = L^3 \)).

(a) Compute the density of states, \( D(E) \), for this system.

Without spin, \( E = \frac{\mathbf{p}^2}{2m} \),\( \Rightarrow \), \( \mathbf{n}^2 = n_x^2 + n_y^2 + n_z^2 \)

\[
D(E) = \frac{\text{volume of sphere of thickness } \Delta n}{\Delta E} = \frac{\frac{\pi}{2} n_x^2 \text{ volume of } \Delta n}{\Delta E} \approx \frac{\pi}{2} \left( \frac{2mL^2 E}{\pi^2 h^2} \right) \sqrt{\frac{2mL^2}{\pi^2 h^2}} \frac{1}{2\sqrt{E}} = \frac{m^{3/2}L^3}{\sqrt{2\pi^3 h^3}} \sqrt{E}.
\]

Fermi sphere:

(b) Using \( D(E) \), write down the expression for the total number of particles \( N \) at \( T = 0 \). Express the Fermi energy \( E_F \) in terms of the particle density, \( n = N/V \).

With spin, multiply by \((2S+1) = 2\):

\[
D(E) = \frac{\sqrt{2}}{2\pi} \frac{m^{3/2}L^3}{h^3} \sqrt{E}.
\]

Now, at \( T = 0 \), \( N = \int_0^\infty dE \, D(E) \, N(E) \) becomes

\[
N = \int_0^{E_F} dE \, D(E) = \frac{\sqrt{2}}{2\pi} \frac{m^{3/2}L^3}{h^3} \frac{2}{3} E_F^{3/2}, \quad \text{or}
\]

\[
E_F = \frac{\pi^2 h^2}{2m} \left[ \frac{3}{\pi} \frac{N}{V} \right]^{2/3} = \left( \frac{3}{\pi} \right)^{2/3} \frac{\pi^2 h^2}{2m} n^{2/3}.
\]
Problem 8

Sketch the band structure for an intrinsic semiconductor, n-type extrinsic semiconductor, and p-type extrinsic semiconductor (clearly label all three cases!). What are the majority carriers in the n-type and p-type semiconductors, and why?

At finite $T$, $\bar{e}$'s from the donor level get excited into the conduction band in n-type SC => the majority carriers are $\bar{e}$'s. Similarly, $\bar{e}$'s from the valence band get excited into the acceptor level. At finite $T$ in p-type SC => the majority carriers are holes in the valence band.
Problem 9 [EXTRA CREDIT]

Hydrogen-like atoms are single-electron atoms with a nuclear charge of \( +Ze \).

(a) Write down the energy levels of a hydrogen-like atom in terms of the energy levels of the hydrogen atom. What is the dependence of energy on \( Z \)?

Recall that in the hydrogen atom,

\[
E_n = -\frac{m e^4}{2 (4\pi \varepsilon_0)^2 \hbar^2} \frac{1}{n^2} = -\frac{13.6 \text{ eV}}{n^2}, \quad n=1, 2, 3, \ldots
\]

So \( \ell^2 \rightarrow Z \ell^2 \), \( E_n = -\frac{13.6 \text{ eV}}{n^2} Z^2 \sim Z^2 \).

Energies are deeper by a factor of \( Z^2 \).

(b) Recall that for \( l = n - 1 \) states of the hydrogen atom, the radial wavefunction is given by

\[
R_{n,n-1}(r) \sim r^{n-1} e^{-r/a_0},
\]

where \( a_0 \) is the Bohr radius.

Compute the radius, \( r_n \), at which the radial probability is maximum for the hydrogen atom. Using this result, find the radius at which the radial probability is maximum for the hydrogen-like atom. What is its dependence on \( Z \)?

The radial probability \( p(r) = r^2 R^2_{n,n-1}(r) \sim r^{2n} e^{-2r/a_0} \)

\[
\frac{dp(r)}{dr} \bigg|_{r_{\text{max}}} = 0 \implies 2n r_{\text{max}}^{2n-1} = \frac{2}{a_0} \quad r_{\text{max}} = 0, \text{ or}
\]

\[
R_{\text{max}} = n^2 a_0.
\]

Here, \( a_0 = \frac{(4\pi \varepsilon_0)^{1/2} \hbar^2}{me^2} \) is the Bohr radius.

Clearly, \( e^2 \rightarrow 2e^2 \) produces

\[
R_{\text{max}} = \frac{n^2 a_0}{2} \sim \frac{1}{2}
\]

The radii are smaller by a factor of \( 2 \).