

## CROSSOVER EXPONENT FOR A WEAK MAGNETIC FIELD AT THE LOCALIZATION ORTHOGONAL FIXED POINT

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We re-examine the non-linear  $\sigma$ -model lagrangian for the soft modes of non-interacting electrons in a disordered system in a weak magnetic field, close to two dimensions. The magnetic field is shown to induce two scaling terms in the low-energy lagrangian. The crossover exponents are computed by using a Wilson-Polyakov non-linear renormalization group. The largest one agrees to lowest order with the exponent expected from naive scaling arguments.

### 1. Introduction

The non-linear  $\sigma$ -model has recently been developed as a powerful tool for analyzing the Anderson model of disordered electronic systems near two dimensions [1-4]. In this paper we apply a non-linear  $\sigma$ -model formalism to compute the crossover exponent in the presence of a weak external magnetic field, a problem first addressed by Khmel'nitsky and Larkin [5]. In the absence of a magnetic field the conductivity  $\sigma$  scales as  $\sigma \sim 1/\xi^{d-2}$  with  $\xi \sim (E - E_c)^{-\nu}$  where  $E_c$  denotes the mobility edge. They argued that a small magnetic field introduces a new length, the "magnetic length",  $L_B = \sqrt{\hbar c/eB}$ , and the conductivity is then a scaling function of  $L_B/\xi$ , i.e.  $\sigma = \xi^{-(d-2)} f(L_B/\xi) = (E - E_c)^{(d-2)\nu} g(B/(E - E_c)^{2\nu})$ , giving a crossover exponent  $\varphi = 2\nu$ . A related problem is the crossover exponent for other time-reversal breaking perturbations. Opperman [6] calculated the crossover exponent for a time-reversal breaking-potential scattering perturbation to be  $\varphi = 2/\varepsilon + O(\varepsilon)$ ,  $\varepsilon = d - 2$ . The same crossover exponent was found for a random, spin-dependent but real, potential. Wegner [7] reconsidered this problem and found a crossover exponent  $\varphi = 2\nu + O(\varepsilon^3)$  for

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both a time-reversal invariance-breaking, spin-conserving random potential and a spin-dependent, time-reversal invariant random potential. However, for a local spin scattering potential he found a different crossover exponent,  $\varphi = 2\nu + 3 + O(\varepsilon^3)$ . More recently, Houghton et al. [8] have examined the case of orthogonal to unitary symmetry crossover induced by the magnetic field. They argued that an arbitrarily weak magnetic field induces an infinite number of relevant operators and therefore the magnetic field crossover problem is more complicated than was previously supposed.

In this paper we examine the non-linear  $\sigma$ -model with a symplectic symmetry group, and find that the magnetic field perturbation is more easily discussed in terms of the vector potential. We compute the crossover exponent using the Wilson–Polyakov [9, 10] renormalization group method applied to the non-linear  $\sigma$ -model and find that to  $O(\varepsilon)$  it is in agreement with the original scaling argument of Khmel'nitsky and Larkin [5]. Moreover, our calculation suggests that the Khmel'nitsky–Larkin result,  $\varphi = 2\nu$ , follows from gauge invariance, and is in fact correct to all orders. In sect. 2 we review the derivation of the non-linear  $\sigma$ -model lagrangian which describes the Anderson transition in a magnetic field, by using the technique of Efetov et al. [2]. The magnetic field induces two terms, the first was identified previously by Efetov et al. and the second is a term which reduces to the topological term identified by Pruisken [4] in the unitary limit. In sect. 3 we compute the recursion relations to order  $\varepsilon$  by using a renormalization group scheme [9, 10], and find the crossover exponents for the magnetic field. Section 4 summarizes the conclusions. The technical aspects of the computations in sect. 3 are relegated to appendix A.

## 2. Derivation of the non-linear $\sigma$ -model

This section contains a brief review of the non-linear  $\sigma$ -model approach to localization as developed by Efetov, Larkin and Khmel'nitsky (ELK) [2] and Pruisken [4]. We introduce a Grassmann representation for the Green-function-generating functional and take the lagrangian to be

$$L[\eta, \eta^*] = i \int d^d r \sum_{a=1}^n \sum_{p=1}^2 \eta_{ap}^*(r) \left[ E - \mathcal{H} + i \frac{\omega}{2} s_p \right] \eta_{ap}(r), \quad (2.1)$$

where

$$\mathcal{H} = \mathcal{H}_0 + V(r), \quad (2.2)$$

$$\mathcal{H}_0 = \frac{1}{2m} \left( \frac{\nabla_\mu}{i} - \frac{e}{c} A_\mu \right)^2, \quad (2.3)$$

$a$  = replica index,  $p$  = energy index and

$$s_p = (-1)^{p+1}, \tag{2.4}$$

and we assume a gaussian-distributed disorder

$$P[V(r)] \propto \exp\left[-\frac{1}{2u^2} \int d^d r V^2(r)\right]. \tag{2.5}$$

We combine the Grassmann fields in pairs to form the basis of a spinorial representation

$$\Psi_{ap} = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta_{ap}^* \\ \eta_{ap} \end{pmatrix}, \tag{2.6}$$

in which the lagrangian can be written as

$$L = i \int d^d r \Psi \cdot \left\{ E - \left[ \frac{1}{2m} \left( \frac{\nabla_\mu}{i} + \frac{e}{c} A_\mu \sigma_3 \right)^2 + V(r) \right] \right\} \Psi + \Psi \cdot \frac{i\omega}{2} \hat{s} \Psi. \tag{2.7}$$

Here the Pauli matrices have the usual action on the spinorial indices and are the identity on replica and energy indices.  $\sigma_3$  in eq. (2.7) ensures that the vector potential  $A$  couples to the current  $-\Psi \cdot \nabla_\mu / i \sigma_3 \Psi = \frac{1}{2}(\eta^* \nabla_\mu / i \eta - \nabla_\mu / i \eta^* \eta)$ . The totally antisymmetric inner product,  $\cdot$ , is given by the charge conjugacy matrix  $\hat{c} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes 1_{ap}$  and  $\hat{s}$  is defined in eq. (2.4). For  $B = 0$  and  $\omega = 0$ ,  $L$  possesses the symplectic symmetry  $Sp(4n)$  which, in the presence of non-zero frequency ( $\omega \neq 0$ ), breaks down to the product group  $Sp(2n) \times Sp(2n)$ . For  $B \neq 0$ , the subgroups of the above groups under which  $L$  remains invariant are isomorphic to  $U(2n)$  and  $U(n) \times U(n)$ , respectively. As usual, the average over gaussian-distributed disorder, performed here via the replica trick, leads to a quartic term which is decoupled via a Hubbard-Stratonovich transform, thereby introducing the  $Q$ -matrix composite fields.  $Q$  obeys  $Q = Q^+$  and  $\hat{c}^T Q^T \hat{c} = Q^+$ . Upon integrating out the spinor fields  $\Psi$ , we are left with a lagrangian for the  $Q$ -fields

$$L[Q] = \frac{1}{2u^2} \int d^d r \text{tr}\{Q^2\} - \frac{1}{2} \int d^d r \text{tr}\left\{\ln\left[E - \mathcal{H}_0 + iQ + i\frac{\omega}{2}\hat{s}\right]\right\}, \tag{2.8}$$

where

$$\mathcal{H}_0 = \frac{1}{2m} \Pi_\mu \Pi^\mu, \tag{2.9}$$

with

$$\Pi^\mu = \left( \frac{\nabla_\mu}{i} + \frac{e}{c} \sigma_3 A_\mu \right), \quad \Pi_\mu = \left( -\frac{\nabla_\mu}{i} + \frac{e}{c} \sigma_3 A_\mu \right). \quad (2.10)$$

The invariance of  $L[\Psi]$  under symplectic transformations  $T \in \text{Sp}(4n)$ ,  $T^T \hat{c} T = \hat{c}$ , implies that the saddlepoint  $Q_{\text{sp}} = \hat{s}/2\tau$  for the above lagrangian is a member of a manifold of saddlepoint solutions  $Q \rightarrow TQ_{\text{sp}}T^{-1}$ . Here  $\tau$  is the scattering time,  $1/\tau = 2\pi u^2 \rho$ ,  $\rho = \rho(E)$  being the density of states. Substituting  $Q = T(\hat{s}/2\tau)T^{-1}$ ,  $T$  slowly varying in  $\text{Sp}(4n)$ , into  $L[Q]$ , and ignoring constant terms, we found

$$L[Q] = -\frac{1}{2} \int d^d r \text{tr} \left\{ \ln \left[ E - T^{-1} \mathcal{H}_0 T + \frac{i\hat{s}}{2\tau} + i\frac{\omega}{2} T^{-1} \hat{s} T \right] \right\}. \quad (2.11)$$

We write

$$T^{-1} \mathcal{H}_0 T = \mathcal{H}_0 + \frac{1}{2m} [D_\mu D^\mu + D_\mu \Pi^\mu + \Pi_\mu D^\mu], \quad (2.12)$$

where  $D_\mu = D^\mu = T^{-1}(\nabla_\mu T/i) + (e/c)T^{-1}[\sigma_3, T]A_\mu$ . We next expand the logarithm to third order in the operator  $D_\mu$  and use the quantity  $\hat{G}^{-1} \equiv E - \mathcal{H}_0 + i(\hat{s}/2\tau)$ .  $\hat{G}$  and  $\mathcal{H}_0$  are diagonal in energy and spinor space. We will use the notation  $\hat{G} = \delta_{\rho\rho'} \delta_{\sigma\sigma'} G_\sigma^\rho$ . There are five terms in the expression of the trace of the logarithm

$$(1) \quad \frac{1}{4m} \text{tr} D_\mu D^\mu \hat{G}, \quad (2.13)$$

$$(2) \quad \frac{1}{4m} \text{tr} D_\mu^{\leftrightarrow} \Pi^\mu \hat{G}, \quad (2.14)$$

$$(3) \quad \frac{1}{4} \left( \frac{1}{2m} \right)^2 \text{tr} D_\mu^{\leftrightarrow} \Pi^\mu \hat{G} D_\nu^{\leftrightarrow} \Pi^\nu \hat{G}, \quad (2.15)$$

$$(4) \quad \frac{1}{6} \left( \frac{1}{2m} \right)^3 \text{tr} D_\mu^{\leftrightarrow} \Pi^\mu \hat{G} D_\nu^{\leftrightarrow} \Pi^\nu \hat{G} D_\rho^{\leftrightarrow} \Pi^\rho \hat{G}, \quad (2.16)$$

$$(5) \quad \frac{1}{2} \left( \frac{1}{2m} \right)^2 \text{tr} D_\mu^{\leftrightarrow} \Pi^\mu \hat{G} D_\nu D^\nu \hat{G}. \quad (2.17)$$

We have defined  $D_\mu^{\leftrightarrow} \Pi^\mu = D_\mu \Pi^\mu + \Pi_\mu D^\mu$ .

Consider first the terms with  $\mu = \nu$ . Terms (1) and (3) combine to give

$$-\frac{1}{8}\sigma_{xx} \text{tr}\{D_\mu \hat{s} D_\mu \hat{s} - D_\mu D_\mu\}, \tag{2.18}$$

after using the Ward identity

$$G^p(r, r) = -\frac{1}{m} \int d^d r' \Pi^x G^p(r, r') \Pi^{x'} G^p(r', r), \tag{2.19}$$

and defining

$$\sigma_{xx} = \frac{1}{dm^2} \int d^d r' \Pi^x G^1(r - r') \Pi^{x'} G^2(r' - r). \tag{2.20}$$

In eqs. (2.19) and (2.20)  $G^p$  indicates the retarded ( $p = 1$ ) and advanced ( $p = 2$ ) component of  $\hat{G}$ . Recalling the definition of the operators  $D_\mu$ , it is straightforward to express eq. (2.18) in terms of  $Q = T\hat{s}T^{-1}$  as

$$\frac{\sigma_{xx}}{16} \text{tr}\left(\left(\nabla Q\right) - i\frac{e}{c}A[Q, \sigma_3]\right)^2. \tag{2.21}$$

Equations (2.16) and (2.17) with  $\mu = \nu = \rho$  contribute terms of order  $A^2\nabla, A\nabla^2$  at least and we neglect them consequently since they are less relevant than the terms of order  $A^2, A\nabla$  which are kept in eq. (2.21).

Consider now the terms with  $\mu \neq \nu$ ; by defining

$$D_\mu = d_\mu + \frac{e}{c}\Sigma_3 A_\mu, \quad d_\mu = T^{-1}\frac{1}{i}\nabla_\mu T, \quad \Sigma_3 = T^{-1}[\sigma_3, T],$$

term (3) splits into two terms and the one proportional to  $d_\mu d_\nu$  can be written as

$$\frac{\sigma_{xy}}{8}\epsilon_{\mu\nu} \text{tr}\{d_\mu d_\nu \hat{s} \sigma_3 + d_\mu \sigma_3 d_\nu \hat{s}\}, \tag{2.22}$$

with

$$\sigma_{xy} = -\frac{1}{m^2} \int d^d r' \Pi^y G_1^2(r - r') \Pi^{x'} G_1^1(r' - r), \tag{2.23}$$

where the upper and lower indices in  $G$  refer to the energy and spinor components, respectively. All the dependence on the magnetic field enters via  $\sigma_{xy}$ , the  $xy$ -component of the conductivity tensor in the Born approximation  $(ne^2\tau^2/m^2c)(\partial_x A_y - \partial_y A_x)$ .

The part of term (3) which is proportional to  $\Sigma_3 A_\nu d_\mu$  is combined with terms (4) and (5) evaluated with  $\hat{G} = \hat{G}_0$ , the Green function in zero field, and  $\Pi^\mu = (1/i)\nabla_\mu$ . To order  $A\nabla^2$ , two operators  $D$  in eq. (2.16) and (2.17) are replaced by  $d$  while the

third D is replaced by  $\Sigma_3 A$ . The result is

$$\frac{\sigma_{xy}}{8} \sum_{p,p'} \left[ (d_x)_{pp'} (d_y)_{p'p} (\Sigma_3)_{pp} - (d_y)_{pp'} (d_x)_{p'p} (\Sigma_3)_{pp} \right] (s_p - s_{p'}), \quad (2.24)$$

where now  $\sigma_{xy}$  is the low-field limit of  $\sigma_{xy}$  in eq. (2.23). Combining eqs. (2.24) and (2.22), the  $\mu \neq \nu$  contributions of terms (3), (4) and (5) add up to

$$\frac{\sigma_{xy}}{8} \sum_{pp'} \left[ (d_x)_{pp'} (d_y)_{p'p} (T^{-1} \sigma_3 T)_{pp} - (d_y)_{pp'} (d_x)_{p'p} (T^{-1} \sigma_3 T)_{pp} \right] (s_p - s_{p'}). \quad (2.25)$$

This is easily rewritten in terms of  $Q$  as

$$\frac{\sigma_{xy}}{16} \epsilon_{\mu\nu} \text{tr} Q \nabla_\mu Q \nabla_\nu Q \sigma_3, \quad (2.26)$$

which generalizes to the present orthogonal case the expression first derived for the unitary case [4]. A naive dimensional analysis shows this term is less relevant than the  $A$ -terms in eq. (2.21). However, because of its possible connection to the Hall conductivity, we shall keep it in the effective lagrangian for the crossover problem in a weak magnetic field.

Combining eqs. (2.26) and (2.21) we then write

$$L[Q] = \frac{\sigma_{xx}}{16} \int d^d r \text{tr} \left( \nabla Q - i \frac{e}{c} A [Q, \sigma_3] \right)^2 + \frac{\sigma_{xy}}{16} \int d^d r \epsilon_{\mu\nu} \text{tr} Q \nabla_\mu Q \nabla_\nu Q \sigma_3. \quad (2.27)$$

That is

$$L = L_{xx} + L_A + L_{A^2} + L_{xy}, \quad (2.28)$$

$$L_{xx} = \frac{\sigma_{xx}}{16} \int d^d r \text{tr} (\nabla Q)^2, \quad (2.29)$$

$$L_A = -i \frac{\sigma_{xx} e}{8c} \int d^d r A_\mu \text{tr} \nabla_\mu Q [Q, \sigma_3], \quad (2.30)$$

$$L_{A^2} = -\frac{\sigma_{xx} e^2}{16c^2} \int d^d r A^2 \text{tr} [Q, \sigma_3]^2, \quad (2.31)$$

$$L_{xy} = \frac{\sigma_{xy}}{16} \int d^d r \epsilon_{\mu\nu} \text{tr} Q \nabla_\mu Q \nabla_\nu Q \sigma_3. \quad (2.32)$$

### 3. Evaluation of the crossover exponents

In this section we use Polyakov's version [10] of Wilson's renormalization group to calculate the crossover exponents of  $L_A$ ,  $L_{A^2}$  and  $L_{xy}$  at the orthogonal fixed point. Following ELK we write  $T = \tilde{T} T_0$  with  $\tilde{T}$  slowly varying (i.e. containing wave vectors  $k < \Lambda/b$ ) and  $T_0$  a fast variable representing modes with wave vectors  $\Lambda/b < k < \Lambda$ . We then integrate out the fast variable  $T_0$  to obtain an effective lagrangian  $\tilde{L}$  for the slow variables  $\tilde{T}$ .

The renormalization of eq. (2.27) with  $A_\mu = \sigma_{xy} = 0$  was considered in ref. [2] where recursion relations for  $\sigma_{xx}$  were derived. They resulted in a fixed point corresponding to the Anderson transition. Here we consider the renormalization of  $L_A$ ,  $L_{A^2}$  and  $L_{xy}$  at this fixed point, to leading order in  $A_\mu$  and  $\sigma_{xy}$  and to one-loop order; this will allow us to derive crossover exponents to lowest order in  $\epsilon = d - 2$ .

To integrate over the fast fields, in the limit of zero replica, it is sufficient to expand  $Q_0 = T_0 \hat{\sigma} T_0^{-1}$  as  $Q_0 = \hat{\sigma} + W$  where  $W = \begin{pmatrix} 0 & \nu \\ \nu^\dagger & 0 \end{pmatrix}$  and  $V$  is a  $n \times n$  matrix ( $n =$  number of replica) of quaternions. Defining  $\tilde{Q} = \tilde{T} \hat{\sigma} \tilde{T}^{-1}$  and  $t^{-1} = \sigma_{xx}/16$  the renormalized  $\tilde{L}_A$  is given (in the limit of zero replica) by

$$\tilde{L}_A = L_A(\tilde{Q}) + \langle L_A^{(1)}(\tilde{T}, W) \rangle_0 - \langle L_A^{(2)}(\tilde{T}, W) \delta L_{xx}(\tilde{T}, W) \rangle_0, \quad (3.1)$$

with

$$\begin{aligned} L_A(\tilde{Q}) &= -i \frac{2e}{ic} \int d^d r A_\mu \text{tr} \nabla_\mu \tilde{Q} [\tilde{Q}, \sigma_3] \\ &= -i \frac{4e}{ic} \int d^d r A_\mu \text{tr} \{ -\hat{\sigma} (\tilde{T}^{-1} \nabla_\mu \tilde{T}) \hat{\sigma} \tilde{T}^{-1} \sigma_3 \tilde{T} + (\tilde{T}^{-1} \nabla_\mu \tilde{T}) \tilde{T}^{-1} \sigma_3 \tilde{T} \}, \end{aligned} \quad (3.2)$$

$$L_A^{(1)} = i \frac{4e}{ic} \int d^d r A_\mu \text{tr} \{ \tilde{T}^{-1} \sigma_3 \tilde{T} W (\tilde{T}^{-1} \nabla_\mu \tilde{T}) \}, \quad (3.3)$$

$$L_A^{(2)} = i \frac{4e}{ic} \int d^d r A_\mu \text{tr} \{ \tilde{T}^{-1} \sigma_3 \tilde{T} W \nabla_\mu W \}, \quad (3.4)$$

$$\delta L_{xx} = \frac{4}{t} \int d^d r \text{tr} \{ (\tilde{T}^{-1} \nabla_\mu \tilde{T}) W \nabla_\mu W \}. \quad (3.5)$$

In eq. (3.1),  $\langle \dots \rangle_0$  denotes the average over  $W$  with the gaussian weight

$$L_0 = \frac{1}{t} \int d^d r \text{tr} \nabla_\mu W \nabla_\mu W = \frac{2}{t} \int d^d r \text{tr} \nabla_\mu V \nabla_\mu V^\dagger.$$

$L_A^{(2)}$  and  $L_A^{(1)}$  are terms of  $L_A(Q)$  which couple  $\tilde{T}$  and  $W$  (with and without gradients of  $W$ , respectively).  $\delta L_{xx}$ , which does contain gradients of  $W$ , originates from  $L_{xx}$ ,  $L_{xx}(Q) = L_{xx}(\tilde{Q}) + L_0(W) + \delta L_{xx}(\tilde{T}, W) + (2/t) \int d^d r \text{tr}(\tilde{T}^{-1} \nabla_\mu \tilde{T}) \times W(\tilde{T}^{-1} \nabla_\mu \tilde{T})W$ .

The rules for carrying out the gaussian averages of eq. (3.1) are discussed in the appendix. Application of this rules gives, in the zero-replica limit,

$$\langle L_A^{(2)} \delta L_{xx} \rangle_0 = 0 \quad \text{and} \quad \langle L_A^{(1)} \rangle_0 = -L_A(\tilde{Q})g, \quad (3.6), (3.7)$$

with

$$g = \int_{\Lambda/b}^\Lambda \frac{d^d q}{(2\pi)^d} \frac{t}{8q^2}.$$

In two dimensions  $g$  reduces to  $g = (t/16\pi) \ln b = (1/\pi \sigma_{xx}) \log b$ . From eqs. (3.1), (3.6) and (3.7) we obtain

$$\tilde{L}_A = L_A(\tilde{Q})(1 - g), \quad (3.8)$$

which states the renormalizability of  $L_A$  at one-loop order with coefficient  $(1 - g)$ .

The renormalized  $\tilde{L}_{A^2}$  is given by

$$\tilde{L}_{A^2} = L_{A^2}(\tilde{Q}) + \langle L_{A^2}^{(1)}(\tilde{T}, W) \rangle_0, \quad (3.9)$$

with

$$\begin{aligned} L_{A^2}(\tilde{Q}) &= -\frac{1}{t} \frac{e^2}{c^2} \int d^d r A_\mu A_\mu \text{tr}[\tilde{Q}, \sigma_3]^2, \\ &= -\frac{2}{t} \frac{e^2}{c^2} \int d^d r A_\mu A_\mu \text{tr}\{\tilde{T}^{-1} \sigma_3 \tilde{T} \hat{s} \tilde{T}^{-1} \sigma_3 \tilde{T} \hat{s} - 1\}, \end{aligned} \quad (3.10)$$

$$L_{A^2}^{(1)} = -\frac{2}{t} \frac{e^2}{c^2} \int d^d r A_\mu A_\mu \text{tr}\{\tilde{T}^{-1} \sigma_3 \tilde{T} W \tilde{T}^{-1} \sigma_3 \tilde{T} W\}. \quad (3.11)$$

In the limit of zero replicas we obtain

$$\langle L_{A^2}^{(1)} \rangle_0 = -L_{A^2}(\tilde{Q})g. \quad (3.12)$$

Equations (3.9) and (3.12) show that  $L_{A^2}$  is renormalized in the zero-replica limit, at one-loop order, with coefficient  $(1 - g)$ , that is

$$\tilde{L}_{A^2} = L_{A^2}(\tilde{Q})(1 - g). \quad (3.13)$$

The above analysis can be generalized at finite number of replicas (see appendix). In this case we find that  $L_A$  is still renormalizable with coefficient  $1 - (1 + 2n)g$ .  $L_{A^2}$  is instead not renormalizable; however, the sum  $L_A + L_{A^2}$  happens to be renormalizable when the analysis of  $L_A + L_{A^2}$  is carried out to second order in  $A$  and we find that

$$\tilde{L}_A + \tilde{L}_{A^2} = [L_A(\tilde{Q}) + L_{A^2}(\tilde{Q})][1 - (1 + 2n)g]. \quad (3.14)$$

Combining eq. (3.14) with the finite replica result [2] for  $L_{xx}$  we obtain

$$\tilde{L}_{xx} + \tilde{L}_A + \tilde{L}_{A^2} = [1 - (1 + 2n)g] \frac{1}{t} \int d^d r \operatorname{tr} \left( \nabla_\mu \tilde{Q} - i \frac{e}{c} A_\mu [\tilde{Q}, \sigma_3] \right)^2, \quad (3.15)$$

as required from gauge invariance. We shall comment on this point in sect. 4.

We turn now to the renormalization of the operator  $L_{xy}$

$$\tilde{L}_{xy} = L_{xy}(\tilde{Q}) + \langle L_{xy}^{(1)}(\tilde{T}, W) \rangle_0 - \langle L_{xy}^{(2)}(\tilde{T}, W) \delta L_{xx}(\tilde{T}, W) \rangle_0. \quad (3.16)$$

The different terms in the right-hand side of this equation are defined as follows

$$L_{xy}(\tilde{Q}) = L_1 + L_2 + L_3 + L_4, \quad (3.17)$$

$$L_1 = \frac{\sigma_{xy}}{16} \epsilon_{\mu\nu} \int d^d r \operatorname{tr} \{ \hat{s} \tilde{d}_\mu \tilde{d}_\nu \tilde{T}^{-1} \sigma_3 \tilde{T} \}, \quad (3.18)$$

$$L_2 = \frac{\sigma_{xy}}{16} \epsilon_{\mu\nu} \int d^d r \operatorname{tr} \{ \tilde{d}_\mu \tilde{d}_\nu \hat{s} \tilde{T}^{-1} \sigma_3 \tilde{T} \}, \quad (3.19)$$

$$L_3 = \frac{\sigma_{xy}}{16} \epsilon_{\mu\nu} \int d^d r \operatorname{tr} \{ \hat{s} \tilde{d}_\mu \tilde{T}^{-1} \sigma_3 \tilde{T} \tilde{d}_\nu \}, \quad (3.20)$$

$$L_4 = \frac{\sigma_{xy}}{16} \epsilon_{\mu\nu} \int d^d r \operatorname{tr} \{ \hat{s} \tilde{d}_\mu \hat{s} \tilde{T}^{-1} \sigma_3 \tilde{T} \hat{s} \tilde{d}_\nu \}, \quad (3.21)$$

while the operators to be averaged in eq. (3.16) are given by

$$L_{xy}^{(1)} = - \frac{\sigma_{xy}}{16} \epsilon_{\mu\nu} \int d^d r \operatorname{tr} \{ W \tilde{d}_\mu W \tilde{d}_\nu \hat{s} \tilde{T}^{-1} \sigma_3 \tilde{T} + W \tilde{d}_\mu \hat{s} \tilde{d}_\nu W \tilde{T}^{-1} \sigma \tilde{T} + \hat{s} \tilde{d}_\mu W \tilde{d}_\nu W \tilde{T}^{-1} \sigma_3 \tilde{T} \}, \quad (3.22)$$

$$L_{xy}^{(2)} = - \frac{\sigma_{xy}}{16} \epsilon_{\mu\nu} \int d^d r \operatorname{tr} \{ \hat{s} \tilde{d}_\mu W \nabla_\nu W \tilde{T}^{-1} \sigma_3 \tilde{T} + W \tilde{d}_\mu \hat{s} \nabla_\nu W \tilde{T}^{-1} \sigma_3 \tilde{T} - W \nabla_\nu W \tilde{d}_\mu \hat{s} \tilde{T}^{-1} \sigma_3 \tilde{T} - \hat{s} \nabla_\nu W \tilde{d}_\mu W \tilde{T}^{-1} \sigma_3 \tilde{T} \}. \quad (3.23)$$

In eqs. (3.18)–(3.23) we have introduced the notation

$$\tilde{d}_\mu = \tilde{T}^{-1} \frac{\nabla_\mu}{i} \tilde{T}.$$

Using the result of appendix A we find that

$$\langle L_{xy}^{(1)} \rangle_0 = (-L_1 - L_2 + L_3 - 3L_4)g, \quad (3.24)$$

$$- \langle L_{xy}^{(2)} \delta L_{xx} \rangle_0 = (-2L_3 + 2L_4)g. \quad (3.25)$$

These equations combined with eqs. (3.16) and (3.17), give

$$\tilde{L}_{xy} = L_{xy}(\tilde{Q})(1-g), \quad (3.26)$$

which shows the renormalizability of  $L_{xy}$  and yields the recursion relation for  $\sigma_{xy}$ .

#### 4. Conclusions

In this paper we have reviewed the derivation of a non-linear  $\sigma$ -model effective lagrangian to investigate the crossover from the  $B = 0$  fixed point in a disordered system of non-interacting electrons in a random potential. Our effective lagrangian in the small magnetic field limit contains three operators: two electromagnetic contributions  $L_A + L_{A^2}$  from the covariant derivative in the quadratic part of the non-linear  $\sigma$ -model first derived by Efetov et al. [2], and a cubic term which reduces to a topological term in models with unitary symmetry [4]. Standard renormalization-group techniques were applied to determine the first-order scaling of these operators. The results indicate that the field-theoretical calculation agrees with previous arguments from naive scaling theory. However, the presence of two scaling behaviors, instead of the single one assumed by the naive scaling theory, may indicate that the scaling analysis of the transport coefficients, as a function of the field, is more complicated than in ref. [5].

We recall that in the limit  $n \rightarrow 0$  the anomalous dimension of the field  $Q$  vanishes. From the recursion relations for  $L_{xx}$  in ref. [2] one finds the recursion relation for  $\sigma_{xx}$ ,

$$\sigma'_{xx} = b^{d-2} \sigma_{xx} \left( 1 - \frac{1}{\pi \sigma_{xx}} \ln(b) \right), \quad (4.1)$$

which yields a fixed point  $\sigma_{xx} = 1/\pi\varepsilon$  and a critical exponent

$$y_l = \nu^{-1} = \varepsilon + O(\varepsilon^2). \quad (4.2)$$

The recursion relation for the operator  $L_A$  involving the vector potential linearly (eq. 3.8), gives:

$$\sigma'_{xx} A'(r) = b^{d-1} \sigma_{xx} \left( 1 - \frac{1}{\pi \sigma_{xx}} \ln b \right) A(br), \tag{4.3}$$

while from eq. (3.13), for  $L_{A^2}$ , we obtain

$$\sigma'_{xx} A'^2(r) = b^d \sigma_{xx} \left( 1 - \frac{1}{\pi \sigma_{xx}} \ln b \right) A^2(br). \tag{4.4}$$

When combined with eq. (4.1), both eqs. (4.3) and (4.4) lead to

$$A'(r) = bA(br). \tag{4.5}$$

This transformation is valid for scales shorter than those over which  $A(r)$  varies. In particular, it is always valid in the limit of a static field  $A_x(q) = (1/2i) B \nabla_{q_y} \delta(q)$ ,  $A_y(q) = -(1/2i) B \nabla_{q_x} \delta(q)$ . Because of  $\nabla_{q_{x,y}}$  the scaling relation for  $B$  acquires an additional factor  $b$ . Then eq. (4.5) gives

$$B' = b^2 B, \tag{4.6}$$

which implies

$$\phi_B = 2\nu. \tag{4.7}$$

This is the microscopic justification of the Khmel'nitzky–Larkin scaling analysis [5]. We suggest that this result for the crossover exponent is exact and follows from gauge invariance. The original lagrangian is in fact invariant under a gauge transformation

$$T \rightarrow e^{i\varphi(r)\sigma_3} T, \quad A \rightarrow A - \frac{c}{e} \nabla\varphi(r).$$

Since the renormalization procedure respects that symmetry, the low-energy lagrangian must obey the same symmetry which implies that the renormalization of  $L_{xx}$  must equal the renormalization of  $L_A + L_{A^2}$  to all orders in  $\varepsilon$ . The present crossover analysis has to be contrasted with the crossover analysis for a time-reversal symmetry-breaking potential [6, 7]. In that case no specific relation exists between the coefficients of the operators  $Q\nabla Q$  and  $QQ$  which are generated by the symmetry-breaking field. The crossover behavior is then determined by the antisymmetric (a) part of the  $QQ$  operator. The  $Q\nabla Q$  term would instead lead to a crossover exponent smaller than the leading crossover exponent by approximately  $\nu$  and is thus subleading. Calling  $\tau_c^{-1}$  the inverse scattering time introduced by the time-reversal symmetry-breaking potential,  $\varphi_{\tau_c^{-1}} = \varphi_a$  coincides with eq. (4.7) at

lowest order in  $\varepsilon$  [6, 7]. However, Wegner [7] recently found corrections to  $\varphi_a$  at four-loop order yielding  $\varphi_a = \nu(2 + \frac{9}{4}\zeta(3)\varepsilon^4 + O(\varepsilon^5))$ . The previous considerations on gauge invariance suggest that these corrections to order  $\varepsilon^4$  do not effect  $L_{\mathcal{A}} + L_{\mathcal{A}^2}$ , so that eq. (4.7) holds to all orders in  $\varepsilon$ .

The renormalization of the second operator induced by the magnetic field (eq. (3.26)) is more mysterious. Here the magnetic field enters only via the transport coefficient  $\sigma_{xy}$  which multiplies an operator which is multiplicatively renormalizable. The recursion relation for its coefficients in the lagrangian is given by

$$\sigma'_{xy} = b^{d-2}\sigma_{xy} \left( 1 - \frac{1}{\pi\sigma_{xx}} \ln(b) \right). \quad (4.8)$$

When linearized around the fixed point it gives the crossover exponent

$$\phi_{\sigma_{xy}} = 0. \quad (4.9)$$

In agreement with naive dimensional analysis  $L_{xy}$  is therefore less relevant than  $L_{\mathcal{A}} + L_{\mathcal{A}^2}$ . However, following the reasoning of refs. [11, 12] it is very likely that  $L_{xy}$  is relevant to the evaluation of the Hall conductivity. If we interpret the renormalization of the operator  $L_{xy}$  in the lagrangian as the renormalization of the physical transport coefficient  $\sigma_{xy}$ , our results would suggest that the Hall conductivity  $\sigma_{xy}$  scales as the conductivity  $\sigma_{xx}$  and therefore the Hall coefficient diverges at the mobility edge. This result agrees with the experimental results of ref. [13]. It supports the notion that the inverse Hall coefficient measures the number of mobile carriers. However, the identification of the renormalization of the operator (2.32) in the lagrangian with the physical conductivity, and the relation of this approach with the weak-scattering perturbation theory of ref. [14], are not as simple as in ref. [12] and will be the subject of a future publication.

It is important to emphasize that there are different operators which break time-reversal invariance and that they are associated with different crossover exponents. In fact the crossover exponents calculated in this paper are different from those considered in ref. [7]. Another physical problem, for which our analysis can be relevant, is the crossover from orthogonal to unitary localization of the acoustic excitations of a superfluid film carrying a uniform current flow [15].

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Appendix A

In this appendix we derive eqs. (3.6), (3.7), (3.12), (3.24) and (3.25) given in the text and used to analyze the renormalization properties of  $L_A$ ,  $L_{A^2}$  and  $L_{xy}$ .

In evaluating  $\langle L_A^{(1)} \rangle_0$ ,  $\langle L_{A^2}^{(1)} \rangle_0$  and  $\langle L_{xy}^{(1)} \rangle_0$  we only need the following basic expression for the gaussian averages

$$\begin{aligned} \langle AW(r_1)BW(r_2) \rangle_0 &= g(r_1 - r_2) \{ \text{tr } A \text{tr } B - \text{tr } \hat{A} \text{tr } \hat{B} \} \\ &\quad - g(r_1 - r_2) \{ \text{tr } A\bar{B} - \text{tr } A\hat{B}\hat{B} \}, \end{aligned} \tag{A.1}$$

where  $\bar{B} = \hat{c}^T B^T \hat{c}$ . Here  $\hat{c} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  is the charge conjugacy matrix in the spinor space and the upper index T indicates the transposed operator.  $g(r)$  is the real correlator whose Fourier transform is given by  $g(q) = t/8q^2$ . Equations (3.6), (3.12), and (3.24) then follow from

$$\bar{d}_\mu = \hat{c}^T \tilde{d}_\mu \hat{c} = -\tilde{d}_\mu, \quad \hat{c}^T (\tilde{T}^{-1} \sigma_3 \tilde{T})^T \hat{c} = -(\tilde{T}^{-1} \sigma_3 \tilde{T}), \tag{A.2}$$

where the identities  $\hat{c}^T (\tilde{T}^{-1})^T \hat{c} = \tilde{T}$ ,  $\hat{c}^T \sigma_3^T \hat{c} = -\sigma_3$  and  $(\nabla_\mu \tilde{T}^{-1}) \tilde{T} = -\tilde{T}^{-1} \nabla_\mu \tilde{T}$  have been used. We note that, because of eq. (A.2),  $\text{tr } \tilde{d}_\mu = \text{tr } \bar{d}_\mu \hat{c} = \text{tr } \tilde{T}^{-1} \sigma_3 \tilde{T} = \text{tr } \tilde{T}^{-1} \sigma_3 \tilde{T} \hat{c} = 0$ .

The derivation of eqs. (3.6) and (3.25) for  $\langle L_{A,xy}^{(2)} \delta L_{xx} \rangle_0$  are more involved. We first note that all terms in  $\langle L_{A,xy}^{(2)} \delta L_{xx} \rangle_0$  can be expressed as double integrals over

$$\nabla_{\mu_2} \nabla_{\mu_4} \langle \text{tr } AW(r_1)BW(r_2) \text{tr } EW(r_3)W(r_4) \rangle_0 |_{(r_1=r_2, r_3=r_4)},$$

with suitable choices of the operators  $A, B$  and  $E = -\bar{E}$ . This correlation function can be readily evaluated by a consecutive application of eq. (A.1) to the following expression

$$\langle \text{tr } AW(r_1) \text{tr } BW(r_2) \rangle_0 = g(r_1 - r_2) \{ \text{tr } AB - \text{tr } A\hat{B}\hat{B} + \text{tr } A\bar{B} - \text{tr } A\hat{B}\bar{B} \}. \tag{A.3}$$

One gets

$$\begin{aligned} &\langle \text{tr } AW(r_1)BW(r_2) \text{tr } EW(r_3)W(r_4) \rangle_0 \\ &= [g(r_2 - r_4)g(r_1 - r_3) - g(r_1 - r_4)g(r_2 - r_3)] \\ &\quad \times \{ [\text{tr } A \text{tr } BE - \text{tr } A\hat{B} \text{tr } BE\hat{B} + \text{tr } A \text{tr } B\hat{E}\hat{E} - \text{tr } A\hat{B} \text{tr } B\hat{E} \\ &\quad - \text{tr } B \text{tr } AE + \text{tr } B\hat{B} \text{tr } AE\hat{B} - \text{tr } B \text{tr } A\hat{E}\hat{E} + \text{tr } B\hat{B} \text{tr } A\hat{E}] \\ &\quad - [-\text{tr } A\bar{E}\bar{B} + \text{tr } A\hat{E}\bar{B}\hat{B} + \text{tr } A\bar{E}\hat{B}\hat{B} - \text{tr } A\bar{B}\hat{E}\hat{B} \\ &\quad - \text{tr } \bar{B}EA + \text{tr } \bar{B}\hat{E}EA\hat{B} + \text{tr } \bar{B}\hat{E}\hat{B}A\hat{B} - \text{tr } \bar{B}A\hat{E}\hat{B}] \}, \end{aligned} \tag{A.4}$$

where  $\bar{E} = -E$  has been used.

To evaluate  $\langle L_A^{(2)}\delta L_{xx}\rangle_0$  we apply eq. (A.4) with  $A = \tilde{T}^{-1}\sigma_3\tilde{T}$ ,  $B = 1$ ,  $E = \tilde{d}_\mu$ . We obtain

$$\langle L_A^{(2)}\delta L_{xx}\rangle_0 = -\frac{4e}{tcd}g \int d^d r A_\mu \left\{ -\text{tr} 1 \left[ \text{tr} \tilde{T}^{-1}\sigma_3\tilde{T}\tilde{d}_\mu + \text{tr} \tilde{T}^{-1}\sigma_3\tilde{T}\hat{s}\tilde{d}_\mu\hat{s} \right] \right\}. \tag{A.5}$$

In deriving eq. (A.5) we have neglected higher order gradient terms (by setting the operators  $\tilde{T}^{-1}\sigma_3\tilde{T}$  and  $\tilde{d}_\mu$  at the same point) and have used the relationship

$$\int d^d r \left[ -(\nabla_x^2 g(r))g(r) + (\nabla_x g(r))(\nabla_x g(r)) \right] = \frac{2}{d} \int \frac{d^d q}{(2\pi)^d} q^2 g^2(q) = \frac{t}{4d}g. \tag{A.6}$$

In the zero-replica limit  $\text{tr} 1 = 4n$  vanishes and eq. (3.6) is recovered.

$\langle L_{xy}^{(2)}\delta L_{xx}\rangle$  can be expressed as the sum of three terms of the form (A.4) with  $E = \tilde{d}_\nu$  and (i)  $A = \epsilon_{\mu\nu}\{\tilde{T}^{-1}\sigma_3\tilde{T}\hat{s}\tilde{d}_\mu - \tilde{d}_\mu\hat{s}\tilde{T}^{-1}\sigma_3\tilde{T}\}$ ,  $B = 1$ ; (ii)  $A = \tilde{T}^{-1}\sigma_3\tilde{T}$ ,  $B = \epsilon_{\mu\nu}\tilde{d}_\mu\hat{s}$ ; and (iii)  $A = -\epsilon_{\mu\nu}\tilde{d}_\mu$ ,  $B = \tilde{T}^{-1}\sigma_3\tilde{T}\hat{s}$ . For the three terms we get

$$\begin{aligned} \text{(i)} \quad & \frac{\sigma_{xy}}{16} \frac{1}{d} g \epsilon_{\mu\nu} \text{tr} 1 \text{tr} \left\{ \tilde{T}^{-1}\sigma_3\tilde{T}\hat{s}\tilde{d}_\mu\tilde{d}_\nu + \tilde{d}_\mu\tilde{d}_\nu\hat{s}\tilde{T}^{-1}\sigma_3\tilde{T} + 2\tilde{T}^{-1}\sigma_3\tilde{T}\hat{s}\tilde{d}_\mu\hat{s}\tilde{d}_\nu\hat{s} \right\} \\ & = \frac{1}{d} g \text{tr} 1 (L_1 + L_2 - 2L_4), \end{aligned} \tag{A.7}$$

$$\text{(ii)} \quad -\frac{\sigma_{xy}}{16} \frac{1}{d} g \epsilon_{\mu\nu} \text{tr} \left\{ 2\tilde{d}_\mu\hat{s}\tilde{d}_\nu\tilde{T}^{-1}\sigma_3\tilde{T} - 2\hat{s}\tilde{d}_\mu\hat{s}\tilde{d}_\nu\hat{s}\tilde{T}^{-1}\sigma_3\tilde{T} \right\} = \frac{1}{d} g (2L_3 - 2L_4), \tag{A.8}$$

$$\text{(iii)} \quad \frac{\sigma_{xy}}{16} \frac{1}{d} g \epsilon_{\mu\nu} \text{tr} \left\{ 2\hat{s}\tilde{d}_\mu\hat{s}\tilde{d}_\nu\hat{s}\tilde{T}^{-1}\sigma_3\tilde{T} - 2\tilde{T}^{-1}\sigma_3\tilde{T}\tilde{d}_\mu\hat{s}\tilde{d}_\nu \right\} = \frac{1}{d} g (-2L_4 + 2L_3). \tag{A.8}$$

By summing the three contributions in the zero-replica limit at  $d = 2$  we obtain

$$\langle L_{xy}^{(2)}\delta L_{xx}\rangle_0 = g(2L_3 - 2L_4), \tag{A.9}$$

which coincides with eq. (3.25) given in the text.

We shall finally discuss the renormalization of  $L_A$  and  $L_{A^2}$  for finite number of replicas. In this case we write down  $L_A^{(1)}$  as

$$L_A^{(1)} = \frac{i4e}{tc} \int d^d r A_\mu \text{tr} \{ \tilde{T}^{-1} \sigma_3 \tilde{T} Q_0 (\tilde{T}^{-1} \nabla_\mu \tilde{T}) Q_0 \}. \quad (\text{A.10})$$

Because  $\langle Q_0 \rangle_0 = \hat{s}(1 - 2ng)$ , we get an additional contribution to be added to eq. (3.7)

$$\langle L_A^{(1)} \rangle_0^{\text{f.r.}} = -4ng \frac{i4e}{tc} \int d^d r A_\mu \text{tr} \{ \tilde{T}^{-1} \sigma_3 \tilde{T} \hat{s} (\tilde{T}^{-1} \nabla_\mu \tilde{T}) \hat{s} \}. \quad (\text{A.11})$$

From eqs. (3.1), (A.5) and (A.11) we obtain the finite replica contribution to  $\tilde{L}_A$

$$\tilde{L}_A^{\text{f.r.}} = -2ng L_A(\tilde{Q}). \quad (\text{A.12})$$

Following the same reasoning we obtain

$$\langle L_{A^2}^{(1)} \rangle_0^{\text{f.r.}} = -4ng \left( -\frac{2}{t} \frac{e^2}{c^2} \right) \int d^d r A_\mu A_\mu \text{tr} \{ \tilde{T}^{-1} \sigma_3 \tilde{T} \hat{s} \tilde{T}^{-1} \sigma_3 \tilde{T} \hat{s} \}. \quad (\text{A.13})$$

To second order in  $A$  and at one-loop order eq. (3.9) has to be corrected by a term

$$-\frac{1}{2} \langle (L_A^{(2)})^2 \rangle_0 = 2ng \left( -\frac{2}{t} \frac{e^2}{c^2} \right) \int d^d r A_\mu A_\mu \text{tr} \{ \tilde{T}^{-1} \sigma_3 \tilde{T} \hat{s} \tilde{T}^{-1} \sigma \tilde{T} \hat{s} + 1 \}. \quad (\text{A.14})$$

Equations (A.13) and (A.14), combined with eqs. (3.9) and (3.10), then yield

$$\tilde{L}_{A^2}^{\text{f.r.}} = -2ng L_{A^2}(\tilde{Q}). \quad (\text{A.15})$$

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