

Relation between the conductivity, the ultrasonic attenuation, and nonlinear σ -model composite operators at the Anderson transition

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We relate the critical behavior of the conductivity and the ultrasonic attenuation close to the Anderson transition to the anomalous dimensions of traceless composite operators introduced by Wegner. Our analysis is used to analyze the scaling behavior of the ultrasonic attenuation coefficient close to the mobility edge.

Noninteracting electrons in a random potential localize when the strength of the disorder exceeds a critical value.¹ This metal-insulator transition, the Anderson transition, is now known to be continuous and is very well understood by analogy with critical phenomena.² The scaling behavior of the conductivity, participation ratio, and dielectric constant near the transition has been analyzed by a variety of renormalization-group techniques.² A most powerful technique is the mapping onto the nonlinear σ model.³ In this Rapid Communication we show that, in the presence of time-reversal symmetry, the anomalous dimensions of the traceless quadratic tensor operators originally introduced by Wegner⁴ in his pioneering work on the participation ratio can be interpreted in terms of the scaling behavior of two transport coefficients, the conductivity and the ultrasonic attenuation, thus relating the latter to the critical behavior of the participation ratio. This relation is surprising since, while the moments of the wave functions are local quantities, the transport coefficients are not easily expressible in terms of local operators.

We start by reviewing the formalism to set up the notation. Products of Green's functions averaged over the disorder can be calculated as expectation values of the following Lagrangian⁵ density:

$$-\mathcal{L}(x) = +\frac{s_p}{2}\phi_p^a\left(H_0 - E + \frac{s_p\omega}{2}\right)\phi_p^a + \frac{\gamma}{8}(\phi_p^a\phi_p^a s_p)^2. \quad (1)$$

$\alpha = 1, \dots, n$ is a replica index; p runs over the values 1, 2 and denotes negative and positive frequencies; s is a diagonal matrix, $s_1 = i$ and $s_2 = -i$. $H_0 = -\nabla^2/2$, ω is the external frequency, and $\gamma = 1/2\pi\rho\tau$ is a measure of the strength of the disorder; $\rho = \rho(E)$ is the density of states and τ the bare scattering time.

Different observables can be expressed as expectation values of the fields ϕ_p^a with respect to Lagrangian Eq. (1). The one-particle Green's function is given by $[G(r, r', \epsilon_p)] = s_p \langle \phi_p^a(r) \phi_p^a(r') \rangle$ with $\epsilon_p = E - s_p\omega/2$; the square brackets denote an ensemble average.

The real part of the conductivity is given by

$$\sigma = \frac{-e^2}{4\pi V} \int d^d x d^d x' \sum_{p,p'} \langle \phi_p^a(x) \nabla_i \phi_p^\beta(x) \phi_p^\beta(x') \nabla_i \phi_p^a(x') \rangle, \quad (2)$$

with V the volume of the system, α, β are two different re-

plicas. The ultrasonic attenuation⁶ coefficient $\tilde{\alpha}(\omega)$ is given by $(\omega^2/2\rho C^3)\alpha(\omega)$. C is the speed of sound and $\alpha(\omega)$ is the stress-stress correlation function

$$\alpha = \frac{1}{4\pi V} \int d^d x d^d x' \times \sum_{p,p'} \langle \nabla_i \phi_p^a(x) \nabla_j \phi_p^\beta(x) \nabla_j \phi_p^\beta(x') \nabla_i \phi_p^a(x') \rangle. \quad (3)$$

To arrive at the nonlinear σ model one decouples the quartic term in Eq. (1) by introducing an order-parameter matrix field $Q_{pp}^{\alpha\beta}(x)$ and integrates out the $\phi_p^a(x)$ field. The resulting Lagrangian is

$$L = \frac{\pi\rho}{8\tau} Q_{pp}^{\alpha\beta} Q_{pp}^{\beta\alpha} + \frac{1}{2} \text{tr} \ln \left[\left(\epsilon_p + \frac{\nabla^2}{2} \right) \delta_{pp'} - \frac{1}{2\tau} Q_{pp}^{\alpha\beta} \right]. \quad (4)$$

The last step in the derivation involves the elimination of the massive modes, i.e., only includes fluctuations of the form given in Eq. (10). The resulting Lagrangian

$$L = \frac{\pi\sigma_0}{8} \text{tr} \nabla Q \nabla Q - \frac{1}{4} \omega \pi \rho \text{tr} s Q, \quad (5)$$

with the constraint $Q^2 = -I$, constitutes the nonlinear σ model.⁵ Here σ_0 is the bare conductivity. In this scheme one obtains $[G(r, r, \epsilon_p)] = \pi\rho \langle Q_{pp}^{\alpha\alpha} \rangle$,

$$[G(r, r', \epsilon_+) G(r', r, \epsilon_-)] = (\pi\rho)^2 \langle Q_{12}^{\alpha\beta} Q_{21}^{\beta\alpha} \rangle,$$

$$[G(r, r\epsilon_+) G(r', r', \epsilon_-)] = (\pi\rho)^2 \langle Q_{11}^{\alpha\beta} Q_{11}^{\beta\alpha} \rangle.$$

The constraint $Q^2 = -I$ is eliminated via the parametrization

$$Q = s^{1/2} T s T^{-1} s^{-1/2}, \quad (6)$$

with T an element of $O(n, n)$.

To find an expression for the conductivity, in terms of the Q variables of the nonlinear σ model, we add a replica-dependent vector potential source term $\frac{1}{2} i a \epsilon_{pp}^{\alpha\beta} \times \phi_p^a(x) \nabla_i \phi_p^\beta(x)$ to the Lagrangian in Eq. (1), $\hat{\epsilon}$ is a matrix obeying $\epsilon_{\alpha\beta}^{\beta\alpha} = -\epsilon_{\alpha\beta}^{\alpha\beta}$ ($\alpha \neq \beta$ being two given replica indices and $|\epsilon_{\alpha\beta}^{\beta\alpha}| = 1$) so that $(e^2/4\pi V)(\partial^2 Z/\partial a^2)$ generates the conductivity in Eq. (2). The introduction of this term is equivalent to a space-dependent rotation of the Q field: $Q \rightarrow (1/\sqrt{s}) U^{-1} \sqrt{s} Q \sqrt{s} U^{-1} (1/\sqrt{s})$, $U = \exp(a X)$ with $X = -i s \hat{\epsilon}$ being a generator of $O(n, n)$.

This transformation applied to Eq. (5), with $\omega=0$, generates

$$L_\sigma = \frac{a^2 \pi \sigma_0}{2} \left(- \sum_{p,p'} (-1)^{p+p'} Q_{pp'}^{\alpha\alpha} Q_{pp'}^{\beta\beta} + \sum_{p,p'} (-1)^{(p+p')} Q_{pp'}^{\alpha\beta} Q_{pp'}^{\beta\alpha} \right) \quad (7)$$

and terms proportional to (∇Q) ,⁷ which will be discussed elsewhere. We obtained the same result by repeating the procedure leading to Eq. (4) in the presence of the source and expanding to second order in a . This methodology will be illustrated in the analysis of the sound absorption.

Equation (7) shows that the conductivity is given by the expectation value of the antisymmetric part, O_A , of the operator considered by Wegner, $v_{ijkl} Q_{ij} Q_{kl}$ with v_{ijkl} obeying the zero trace condition $\sum_i v_{ijil} = \sum_i v_{iikl} = 0$. The index i combines a replica and energy index

$$O_A = \frac{1}{4} \left(\sum_{p,p'} Q_{pp'}^{\alpha\beta} Q_{pp'}^{\beta\alpha} (-1)^{p+p'} - \sum_{p,p'} Q_{pp'}^{\alpha\alpha} Q_{pp'}^{\beta\beta} (-1)^{p+p'} \right). \quad (8)$$

Standard renormalization-group analysis gives $\sigma = \sigma_0 \langle O_A \rangle \sim (E - E_c)^{x_A \nu}$ with x_A the anomalous dimension of the antisymmetric operator, $E - E_c$ the distance from the mobility edge, and ν the localization length exponent $\xi \sim (E - E_c)^{-\nu}$. Wegner's calculation^{4,8} gives $\nu x_A = 1 + O(\epsilon^3)$, in agreement with previous results.² The physical meaning of the Eq. (7) can be elucidated by rewriting the corrections to the conductivity in Eq. (2) in momentum space and separating different regions of small momentum transfer:

$$\begin{aligned} \delta\sigma &\sim \sum_p \sum_{p'} pp' [\phi_{\mp}^{\alpha}(p) \phi_{\pm}^{\beta}(-p) \phi_{\mp}^{\alpha}(p') \phi_{\pm}^{\beta}(-p')] \\ &\approx \sum_{q,p} p(p+q) \phi_{\mp}^{\alpha}(p) \phi_{\pm}^{\beta}(-p) \phi_{\mp}^{\alpha}(p+q) \phi_{\pm}^{\beta}(-p-q) \\ &\quad + \sum_{q,p} p(-p+q) \phi_{\mp}^{\alpha}(p) \phi_{\pm}^{\beta}(-p) \phi_{\mp}^{\alpha}(-p+q) \phi_{\pm}^{\beta}(p-q). \end{aligned}$$

Since p is a large momentum transfer it is natural to identify

$$\begin{aligned} Q_{\mp}^{\alpha+}(q) &\sim \phi_{\mp}^{\alpha}(p) \phi_{\mp}^{\alpha}(p+q); \\ Q_{\pm}^{\beta-}(-q) &\sim \phi_{\pm}^{\beta}(-p) \phi_{\pm}^{\beta}(-p-q), \\ Q_{\mp}^{\alpha-}(-q) &\approx \phi_{\mp}^{\alpha}(p) \phi_{\pm}^{\beta}(+p-q); \\ Q_{\mp}^{\beta-}(q) &\approx \phi_{\mp}^{\alpha}(-p+q) \phi_{\pm}^{\beta}(-p), \end{aligned}$$

with p averaged over the Fermi surface. We then find

$$\delta\sigma \sim \sum_q \langle Q_{\mp}^{\alpha+}(q) Q_{\pm}^{\beta-}(-q) - Q_{\mp}^{\alpha-}(-q) Q_{\mp}^{\beta-}(q) \rangle. \quad (9)$$

This is the part of the operator derived in Eq. (7) which gives rise to singular terms in perturbation theory. $\delta\sigma$ is therefore the expectation value of a local operator, and we then see that the different parts of this operator refer to different regions of singular momentum transfer with a sign that depends on the product of the vertices.⁹

To derive the form of the local operator representing the

ultrasonic attenuation, we add a source $\frac{1}{2} i a \epsilon_{pp'}^{\alpha\beta} \phi_p^{\alpha}(x) \nabla_i \nabla_j \times \phi_p^{\beta}(x)$, $\hat{\epsilon} = \epsilon_{pp'}^{\alpha\beta}$ being now a symmetric matrix, $\epsilon_{pp'}^{\alpha\beta} = \epsilon_{p'p}^{\beta\alpha}$. We identify the term proportional to a^2 in the nonlinear σ model by repeating the steps which lead to Eqs. (4) and (5) in the presence of the source. We insert in Eq. (4) the expression (6) for Q , and neglecting terms proportional to $a\omega Q$ and $a\nabla Q$ we find the additional contribution

$$\frac{1}{2} \text{tr} \ln \left[1 + i a T^{-1} s^{-1} \hat{\epsilon} T \nabla_i \nabla_j \frac{1}{(\epsilon_p + \nabla^2/2) - s/2\tau} \right]. \quad (10)$$

Expanding to second order we find

$$L_u = \frac{a^2 \pi \alpha_0}{2} \int d^d x \left(- \sum_{p,p'} Q_{pp'}^{\alpha\beta} Q_{pp'}^{\beta\alpha} (-1)^{p+p'} - \sum_{p,p'} Q_{pp'}^{\alpha\alpha} Q_{pp'}^{\beta\beta} (-1)^{p+p'} \right), \quad (11)$$

where α_0 is the bare value of α . By taking $(-1/4\pi V) \times (\partial^2 Z / \partial a^2)$ we identify $\alpha = \alpha_0 \langle O_u \rangle$ with

$$O_u = \frac{1}{4} \left(- \sum_{p,p'} Q_{pp'}^{\alpha\beta} Q_{pp'}^{\beta\alpha} (-1)^{p+p'} - \sum_{p,p'} Q_{pp'}^{\alpha\alpha} Q_{pp'}^{\beta\beta} (-1)^{p+p'} \right). \quad (12)$$

To motivate this result physically we apply the heuristic argument following Eq. (8). Analysis of the regions of small momentum transfer suggests that the singular corrections to the transverse ultrasonic attenuation are given by

$$\delta\alpha \sim - \langle [Q_{\mp}^{\alpha+} Q_{\pm}^{\beta-} + Q_{\mp}^{\alpha-} Q_{\mp}^{\beta-}] \rangle. \quad (13)$$

The relative minus sign in Eq. (13), relative to Eq. (9), stems from the fact that the stress vertex $p_x p_y$ does not change sign under backscattering while the current vertex p_x does. The $Q_{+-} Q_{+-}$ which gives the contribution of the region $p = -p'$ is therefore weighted with a different sign in the conductivity and ultrasound calculation.

Results from Eq. (12) can be used to analyze the critical behavior of $\alpha(\omega)$ at the mobility edge. It was conjectured on physical grounds by Kotliar and Ramakrishnan¹⁰ that α should be anomalously enhanced and should exhibit scaling behavior close to the mobility edge. They proposed $\alpha \sim \xi^\epsilon$ and $\alpha \sim \omega^{-\epsilon/2}$ and extracted the exponent from a direct exponentiation of the perturbative series. Kirkpatrick and Belitz¹¹ showed that this naive exponentiation is inconsistent with higher-order terms in perturbation theory which they calculated.

Our analysis clarifies this point and allows us to extract the critical behavior of $\alpha(\omega)$ at the mobility edge. According to Wegner, operator Eq. (12) has to be decomposed into an antisymmetric part, defined in Eq. (8), and a symmetric part

$$O_S = \frac{1}{4} \left(- 2 \sum_{p,p'} Q_{pp'}^{\alpha\beta} Q_{pp'}^{\beta\alpha} (-1)^{p+p'} - \sum_{p,p'} Q_{pp'}^{\alpha\alpha} Q_{pp'}^{\beta\beta} (-1)^{p+p'} \right). \quad (14)$$

Unlike the conductivity, Eq. (12) has components on the

symmetric part which is characterized by an exponent x_s , which is negative and is therefore more relevant than the antisymmetric part. Its decomposition into its symmetric and antisymmetric components is given by

$$O_u = \frac{2}{3}O_S + \frac{1}{3}O_A . \quad (15)$$

Wegner showed^{4,8} that $\langle O_S \rangle \sim (E - E_c)^{\nu x_s}$, with $\nu x_s = -2 + O(\epsilon^3)$.

At the mobility edge, as a function of frequency, $\langle O_A \rangle \sim \omega^{\epsilon/d}$ and $\langle O_S \rangle \sim \omega^{-2\epsilon/d}$. At the mobility edge the symmetric operator controls the critical behavior, and $\alpha(\omega) = \omega^\zeta$ with

$$\zeta = \frac{-2\epsilon}{2+\epsilon} + O(\epsilon^4) . \quad (16)$$

Away from criticality

$$\alpha \sim \xi^{2\epsilon + O(\epsilon^3)} . \quad (17)$$

Therefore, the conjectured exponent in Ref. 10 is incorrect by a factor of 2. Equation (15) explains why the logarithmic series which arises in the perturbation calculation of

the ultrasonic attenuation does not exponentiate.¹¹ The perturbation terms of O_S and O_A exponentiate separately but the series for $O_u = \frac{2}{3}O_S + \frac{1}{3}O_A$, of course, does not. In fact, combining Eq. (15) with Wegner results we find, by expanding the exponentials in logarithmic series and keeping $\ln(\omega\tau)$ and $[\ln(\omega\tau)]^2$ terms to lowest order in ϵ ,

$$\alpha = 1 - \epsilon/2[\ln(\omega\tau)] + (3\epsilon^2/8)[\ln(\omega\tau)]^2 , \quad (18)$$

which is the perturbative result of Kirkpatrick and Belitz.¹¹

The ultrasonic attenuation is a measure of an effective electron-phonon coupling. Our analysis reveals that this coupling is enhanced and the enhancement is proportional to the participation ratio⁴ as one would expect on physical grounds.

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