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One-dimensional spin-glass model with long-range random interactions

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We consider an Ising chain with Hamiltonian $H=\sum_{i>j}(\epsilon_{ij}S_iS_j)/(a|i-j|)^\sigma$, where the ϵ_{ij} are independent random variables. We find a phase transition for $\frac{1}{2}<\sigma<1$. For $\frac{1}{2}<\sigma<\frac{2}{3}$ the critical exponents exhibit mean-field classical behavior. Near $\sigma=1$ we find a smoothly varying specific heat. We investigate the critical behavior near the upper and lower critical range by means of an ϵ expansion around $\sigma=1$ and $\frac{2}{3}$.

One-dimensional spin models with long-range interactions exhibit nontrivial critical behavior and may serve as instructive analogies of more physical, higher-dimensional models. One well-known example is the $r^{-\sigma}$ ferromagnetic chain,[1] which is disordered when $\sigma>2$ but exhibits a phase transition when $1<\sigma\leqslant 2$. The case $\sigma=2$ corresponds to a system at its lower critical dimensionality and is of particular importance in the Kondo problem,[2] and served as a model for the Kosterlitz-Thouless theory of phase transitions.

In this Communication we consider the corresponding chain with long-range, random sign interactions. This model contains both frustration and disorder, and may illustrate many properties of more realistic three-dimensional spin-glass models for which no satisfactory theory yet exists.[3] Below the lower critical range $(\sigma=\frac{2}{3})$ the model is expected to have mean-field behavior and may offer additional insight into the well-studied Sherrington-Kirkpatrick model.[4-7] More important, for σ near the upper critical range $(\sigma=1)$, this model may elucidate the properties of short-ranged spin-glass models near their lower critical dimensionality, where very little is known. This work may be viewed as a first step in this direction.

We consider a one-dimensional system with Hamiltonian

$$H=\sum_{i<j}\frac{\epsilon_{ij}S_iS_j}{|a(i-j)|^\sigma},\quad \frac{1}{2}<\sigma<\frac{3}{2}\ , \tag{1}$$

where the S_i are Ising spins, a is the lattice spacing, and ϵ_{ij} are identically distributed independent Gaussian random variables with probability distribution

$$P(\epsilon_{ij})=\frac{1}{(2\pi J^2)^{1/2}}\exp(-\epsilon_{ij}^2/2J^2)\ .$$

The high-temperature expansion of the free energy is

$$\begin{aligned}-\beta F&=\left\langle \ln\mathrm{Tr}\prod_{i<j}\exp\left(\frac{\beta\epsilon_{ij}S_iS_j}{|a(i-j)|^\sigma}\right)\right\rangle_{\mathrm{av}}\\&=\sum_{i<j}\ln\cosh\frac{\beta\epsilon_{ij}}{|a(i-j)|^\sigma}+N\ln 2\\&\quad+\ln 2^{-N}\mathrm{Tr}\prod_{i<j}S_iS_j\tanh\frac{\beta\epsilon_{ij}}{|i-j|^\sigma}\ ,\end{aligned} \tag{2}$$

where $\langle f\rangle_{\mathrm{av}}\equiv\int f(\epsilon_{ij})P(\epsilon_{ij})\,d\epsilon_{ij}$. To estimate the singular contributions we sum polygon diagrams with adjacent vertices joined by two lines. This results in

$$-\beta F=N\ln 2+\frac{N}{2}\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}e^{-x^2/2}\sum_{j=1}^{\infty}\ln\cosh\frac{\beta Jx}{|ja|^\sigma}-\frac{1}{2}\sum_{n=3}^{\infty}1/2n\sum_{j_0j_1\cdots j_{n-1}}g(j_0-j_1)g(j_1-j_2)\dots g(j_{n-1}-j_0)\ , \tag{3}$$

where

$$g(j)=\langle\tanh^2(\epsilon_{ij}\beta J/|aj|^\sigma)\rangle_{\mathrm{av}}\simeq(J\beta)^2/(2ja)^{2\sigma}$$

for large j. Its Fourier transform is given by

$$g(\kappa)\equiv\sum_{n=-\infty}^{\infty}g(n)e^{in\kappa a}\simeq 2(J\beta)^2[\zeta(2\sigma)+\Gamma(1-2\sigma)\sin(\sigma\pi)(\kappa a)^{2\sigma-1}]$$

for small κ. Then

$$-\beta F\simeq N\left(\ln 2+\frac{(\beta J)^2}{2}\sum_{n=1}^{\infty}\frac{1}{|na|^{2\sigma}}\right)+\frac{N}{4}\int_{-\pi/a}^{\pi/a}\frac{d(\kappa a)}{2\pi}\ln[1-g(\kappa)]\ . \tag{4}$$

The free energy (4) converges for $\sigma > \frac{1}{2}$, as shown rigorously in Ref. 8. F is analytic in T at high temperatures. However, a weak singularity appears when β reaches $\beta_c = 1/J[2\zeta(2\sigma)]^{1/2}$. The diagrams used in this calculation are those considered by Thouless *et al.*[9] In the same approximation we calculate the Fisch-Harris nonlocal order parameter[10]

$$Q_{lj} = \langle\langle S_l S_j \rangle^2\rangle_{\rm av} = \sum_\kappa a(\kappa) e^{i\kappa(l-j)} \ .$$

Adding up the chain diagrams with two lines linking adjacent points, we find

$$Q(\kappa,T) = \frac{g(k)}{1-g(\kappa)} \simeq (T_c/T)^2/[1-(T_c/T)^2 - 2\Gamma(1-2\sigma)\sin\sigma\pi(J/T_c)^2(\kappa a)^{2\sigma-1}] \tag{5}$$

for small κ. When $T > T_c$, Q_{lj} decays exponentially. The nonlinear susceptibility $Q(\kappa=0)$ diverges as $T \to T_c$.[11] For $\sigma \geq 1$ the integral $\int d\kappa\, e^{i\kappa x} Q(q, T=T_c)$ diverges, suggesting that fluctuations destroy any type of ordering. The analysis below cannot predict the precise form of the singularities in the critical region. However, it indicates the existence of a phase transition when $\frac{1}{2} < \sigma < 1$. Renormalization-group techniques are needed to analyze the critical behavior of the model.

When $\sigma = \frac{1}{2}$ the thermodynamical limit of the free energy does not exist; the interaction energy grows faster than the volume. Near that limit one would naively expect mean-field theory to be exact. This argument can be made more rigorous by a renormalization-group analysis. The d-dimensional generalization of Eq. (1) is

$$H = \frac{1}{2} \sum_{x \neq x'} \frac{\epsilon_{xx'} S_x S_{x'}}{|(x-x')a|^\sigma}, \quad \frac{d}{2} < \sigma < \frac{d+2}{2}, \quad x,x' \in Z^d, \quad d \leq 6 \ . \tag{6}$$

The Landau-Ginzburg-Wilson effective Hamiltonian corresponding to Eq. (6) can be written using replicas[12,13] in real and momentum space as

$$H(Q_{\alpha\beta}) = \frac{1}{4} \sum_q [q^{2\sigma-d} \mathrm{Tr}\, Q_{\alpha\beta}(q)\, Q_{\beta\alpha}(-q)] - \frac{w}{(Na^d)^{1/2}} \sum_{q_1,q_2,q_3} Q_{\alpha\beta}(q_1)\, Q_{\beta\gamma}(q_2)\, Q_{\gamma\alpha}(q_3)\, \delta_{q_1+q_2+q_3} \ , \tag{7}$$

$$Q_{\alpha\beta} \equiv \langle S_\alpha S_\beta \rangle; \ \alpha,\beta = 1, \ldots, n; \ n \to 0 \ .$$

We omitted quartic terms and a $q^2 Q_{\alpha\beta}(q)\, Q_{\beta\alpha}(-q)$ term in Eq. (7) since they are irrelevant near $\sigma = 2d/3$. A derivation of Eq. (7) for short-range interactions is given in Ref. 13. The Hamiltonian (6) generates a $q^{2\sigma-d} \mathrm{tr}\, Q(x)\, Q(-x)$ term, which reflects the long-range character of the interaction, as well as a short-range term $q^2 \mathrm{tr}\, Q(-q)\, Q(q)$. In the disordered phase replica symmetry is not broken. The renormalization-group (RG) recursion relations for the couplings r,w are derived as in Ref. 12:

$$r' = b^{2-\eta}\left[r - 36(n-2)w^2 \int_{1/b}^1 \frac{dq' q'^{d-1} \kappa_d}{(q'^{2\sigma-d}+r)^2}\right] \ , \tag{8}$$

$$w' = b^{(6-3\eta-d)/2}\left[w + 36(n-2)w^3 \int_{1/b}^1 \frac{dq' q'^{d-1} \kappa_d}{(q')^{3(2\sigma-d)}}\right] \ , \tag{9}$$

where $(2\pi)^d \kappa_d$ is the solid angle subtended by the sphere in d dimension. Since the renormalization does not generate new $q^{2\sigma-d} Q(q)\, Q(-q)$ terms, $\eta = 2+d-2\sigma$ to all orders. This is a general feature of long-range couplings.[14] From Eqs. (8)–(9) we see that the cubic term is irrelevant around $w=0$ for $\sigma < 2d/3$, so that the critical behavior is Gaussian and mean-field theory is valid for $d/2 < \sigma < 2d/3$. The critical exponents are

$$\nu = 1/(2\sigma - d), \quad \eta = d+2-2\sigma \ . \tag{10}$$

For $\sigma > 2d/3$ the cubic term becomes relevant. Expanding in $\sigma = (2d+\epsilon)/3$ we find a new fixed point at $r^* = -3\epsilon/d$, $w^{*2} = (\epsilon/72)\kappa_d$, with a critical exponent $\nu = 3/(d-4\epsilon)$.

The previous analysis is restricted to a neighborhood of $\sigma = 2d/3$, but is valid in all dimensions. To investigate the critical behavior of (1) near $\sigma = 1$ we use a different renormalization group which exploits the unidimensionality of the model. Applying the replica method to the Hamiltonian (1) we obtain

$$\exp(-H_n[\sigma_i^\alpha]) \equiv \int P(\epsilon_{ij})\, d\epsilon_{ij} \exp\left[\beta \sum_{i\neq j} \sum_\alpha \epsilon_{ij} \sigma_i^\alpha \sigma_j^\alpha / (a|i-j|)^\sigma\right] = \exp \sum_{i \neq j} \frac{\beta^2 J^2}{|a(i-j)|^{2\sigma}} \left[\sum_\alpha \sigma_i^\alpha \sigma_j^\alpha\right]^2 \ . \tag{11}$$

The average free energy is given by[15]

$$-\langle\beta F\rangle_{\rm av}=\lim_{n\to 0}\frac{1}{n}\ln(\mathrm{tr}\,e^{-\beta H_n[\sigma]})\ .$$

At each site i we introduced variables $\sigma_i^\alpha=\pm 1$, $\alpha=1\ldots n$. The possible values of σ_i describe an n-dimensional hypercube with center at the origin. The variables σ_i can take 2^n different values S_α, $\alpha=1\ldots 2^n$. We rewrite the partition function for the Hamiltonian of Eq. (13) in terms of defect variables, and find renormalization-group equations for the defect fugacities and coupling constants.[16]

The interaction energy between a spin in state α at site i and one in state β at site j is

$$\frac{\kappa(\alpha,\beta)}{(a|i-j|)^{2\sigma}}\equiv\frac{\beta^2J^2[(S_\alpha S_\beta)^2-n^2]}{(a|i-j|)^{2\sigma}}\ , \tag{12}$$

where a constant has been subtracted from H_n in Eq. (13) in order that $\kappa_{\alpha\alpha}=0$. We denote by $Y_{\alpha\beta}$ the fugacity of an $\alpha\beta$ defect. Note that $\kappa_{\alpha\beta}$ and $Y_{\alpha\beta}$ depend only on the relative position of S_α and S_β in the hypercube[17] [i.e., $\kappa(\pi\alpha,\pi\beta)=\kappa(\alpha,\beta)$, $Y_{\pi\alpha,\pi\beta}=Y_{\alpha\beta}$ if π is any symmetry of the hypercube]. The partition function in terms of kink variables, assuming periodic boundary conditions, is given by

$$\begin{aligned}Z_n=\sum_{n=0}^{\infty}\sum_{\alpha_1\ldots\alpha_n}Y_{\alpha_1\alpha_2}\cdots Y_{\alpha_n\alpha_1}\int[dr_1\ldots dr_n/a]\\ \times\prod_i\Theta(r_{i+1}-r_i-a)\exp\Bigg\{-1[2(1-\sigma)(2\sigma-1)]\\ \times\sum_{m<n}[\kappa(\alpha_{m+1}\alpha_n)+\kappa(\alpha_m\alpha_{n+1})-\kappa(\alpha_m\alpha_n)-\kappa(\alpha_{m+1}\alpha_{n+1})]\\ \times\left[\left(\frac{r_m-r_n}{a}\right)^{2-2\sigma}-1\right]\Bigg\}\ .\end{aligned} \tag{13}$$

The variables α_i run over spin states on the hypercube, and $Y_{\alpha\alpha}\equiv 0$. A change in the lattice spacing $a\to ae^l$ can be compensated by a change in the kink fugacities and coupling constants which leaves the partition function invariant. This leads to the renormalization-group equations

$$\frac{d}{dl}Y_{\alpha\beta}=Y_{\alpha\beta}\left(1+\frac{2\kappa(\alpha,\beta)}{(2\sigma-1)}\right)+\sum_{\nu\neq\alpha,\beta}Y_{\alpha\nu}Y_{\nu\beta}\ , \tag{14}$$

$$\frac{d}{dl}\kappa(\alpha,\beta)=\kappa(\alpha,\beta)(2-2\sigma)-\sum_\nu Y_{\alpha\nu}^2[\kappa(\alpha,\beta)-\kappa(\beta,\nu)+\kappa(\alpha,\nu)]-\sum_\nu Y_{\beta\nu}^2[\kappa(\alpha,\beta)-\kappa(\alpha,\nu)+\kappa(\beta,\nu)]\ . \tag{15}$$

The derivation of Eqs. (13)–(15) is given in Ref. 16 for the case of logarithmic interactions. The extension to power-law interactions[18] does not pose new problems. For $\sigma=1$ Eqs. (14) and (15) reduce to the results of Ref. 16. For $n=2$, where there are only two states, our model reduces to the Ising model; the nonlinear term in the fugacities vanish and our equations reduce to those in Refs. 2 and 18. To obtain useful results for the spin-glass problem we must analytically continue Eqs. (14) and (15) to $n\to 0$. These equations are exact for $\sigma=1-\epsilon/2$ and small fugacities, and have a symmetric fixed point with $Y_{\alpha\beta}^2=O(\epsilon)$,

$$Y_{\alpha\beta}=Y^*=(\epsilon/2^{n+1})^{1/2}+O(\epsilon)\ , \tag{16}$$

$$\kappa(\alpha,\beta)=\kappa^*=\tfrac{1}{2}[-1+(2-2^n)(\epsilon/2^{n+1})^{1/2}]+O(\epsilon)\ . \tag{17}$$

Linearizing Eqs. (18) and (19) around the symmetric fixed point, we find to first order in ϵ

$$\delta\dot\kappa(\alpha,\beta)=4Y^*\delta Y_{\alpha\beta}+2Y^*\sum_{\nu\neq\beta}\delta Y_{\alpha\nu}\ , \tag{18}$$

$$\delta\dot Y_{\alpha\beta}=(2-2^n)Y^*\delta Y_{\alpha\beta}+2Y^*\delta\kappa(\alpha,\beta)+2Y^*\sum_{\nu\neq\beta}\delta Y_{\alpha\nu}\ . \tag{19}$$

The linearized renormalization-group matrix is a $2(2^n-1)\times 2(2^n-1)$ matrix. Fortunately, due to its high degree of symmetry we can deduce the form of its eigenvectors and perform the analytic continuation of the eigenvalues to $n=0$. In this limit only one relevant eigenvalue $1/\nu$ exists.[19] The eigenvalues are $1/\nu=1.1\sqrt{\epsilon}$ and $\lambda_2=-1.8$. From the scaling relations we find a specific-heat exponent $\alpha=2-\nu=2-0.9/\sqrt{\epsilon}$. Near $\sigma=1$, $\alpha\simeq-\infty$ and the specific heat is smooth at T_c.

These results demonstrate that an Ising chain with long-range random interactions has many properties one expects to find in more realistic spin-glass models, with the range of the interactions $1/\sigma$ playing the roll of dimensionality. For $\frac{1}{2} \leq \sigma \leq \frac{2}{3}$ (high dimensionalities) mean-field theory is correct. As σ increases (d decreases), there is a dramatic departure from mean-field behavior as exemplified by the smoothness of the specific heat as σ approaches 1. For $\sigma > \frac{3}{2}$, i.e., for even lower dimensionalities, the interactions become so weak that they are amenable to rigorous treatment.[20,21] There is no symmetry breaking[21] but a weak Griffith singularity makes the free energy nonanalytic,[20] a phenomenon occurring in many low-dimensional random magnets.[22] It is amusing to speculate that the lower critical dimensionality can be found by equating $d^d\kappa/\kappa^2 = d\kappa/\kappa^{\sigma-1}$; this gives $d=3$ for the lower critical dimensionality of the nearest-neighbor, Ising spin-glass.

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