ONE DIMENSIONAL RANDOM SYSTEMS WITH LONG RANGE INTERACTIONS

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A thesis
presented to Princeton University
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
in
Physics

Princeton, New Jersey, 1983

ABSTRACT

We consider two one dimensional models of disorder.

A percolation model is defined on a one dimensional chain. The bond occupation probability depends on the bond length as

$$P(i,j) = 1 - e^{-\frac{|K|}{[i-j]^{\sigma}}}$$

A percolation threshold in K exists for $1 \leq \sigma \leq 2$. The critical behaviour is analysed using a renormalization group developed for this problem.

The spin glass model is defined by an Ising chain with hamiltonian

$$H = \sum_{i>j} \frac{J_{ij} S_i S_j}{|i-j|^{\sigma}}$$

The J_{ij} are Gaussian random variables with zero mean and variance J^2.

We find a phase transition for $\frac{1}{2} < \sigma < 1$. For $1 < \sigma < \frac{2}{3}$ the system has classical critical behaviour. Near $\sigma = 1$ we find a smooth specific heat. We investigate the critical behaviour near the upper and lower critical range using the renormalization group and an ε expansion around the lower and upper critical range, $\sigma = \frac{2}{3}$ and $\sigma = 1$.

Chapter I

INTRODUCTION

In recent years we have seen considerable progress in the theoretical understanding of amorphous materials. It has become clear by now that random systems are in many ways essentially different from the much better understood, pure, translational invariant ones. New problems demand new concepts and a whole new set of theoretical ideas like percolation, localization, frustration and broken ergodicity are emerging as essential for the understanding of disordered media.

A first example of a physical phenomenon characteristic of a random system and having no analog in pure substances is the phenomenon of percolation. In the percolation problem one has a set of bonds that can be closed or open with a certain probabilty p. If the probabilty p is very small most of the bonds are open and the system is disconnected. For p greater than a critical value, there are enough closed bonds for the formation of an infinite custer of connected sites.

Percolation is the process of formation of an infinite connected cluster. There are no dynamical variables that play a role in the transition, and it is often referred as a geometrically driven phase transition. Other transitions in this class are the gelation and the vulcanization of polymers. Lacking a natural order parameter (like the magnetization in a ferromagnet) they pose new challenges to the theorist interested in phase transitions and they have been the subject of intensive research in the last few years.

A second model of disorder is the spin glass, which models impurity spins interacting via exchange couplings of random sign. At low temperatures the spins freeze in random orientations. The kind of broken symmetry is inextricably linked with the fact that the system is random, and the nature of the order parameter describing the transition is still controversial.

Randomness adds a whole new dimension to the types of critical behaviour that can occur in the neighbourhood of a continuous transition. In the renormalization group language, phase transition in random systems are in different universality classes than their pure counterparts. In this thesis we study two new models of randomness. A one dimensional model of percolation, and a one dimensional spin glass.

It is well known that one dimensional systems with short range interactions cannot exhibit phase transitions at finite temperature. In order to obtain nontrivial critical behaviour we will consider power law interactions. Both models are "random relatives" of the Ising chain with $\frac{1}{r^\sigma}$ interactions. These models have an interesting history. They were first studied by Dyson as nontrivial examples of systems exhibiting phase transitions. The $\sigma=2$ case was studied in detail by Anderson and Yuval in connection with the Kondo problem. The existence of a phase transition in the latter case was recently established "rigorously" by Frolich and Spencer.

The parameter σ plays the role of dimensionality. It controls the density of low energy excitations and the strength of the fluctuations as well. The renormalization group has taught us to look at dimensionality as a continuously variable parameter. There exist at least two critical dimensionalities, an upper critical dimensionality above which fluctuations do not affect the critical behaviour, and a lower critical dimensionality below which the system ceases to have a phase transition. Critical exponents can sometimes be calculated in an ε expansion around these dimensionalities.

Pushing the analogy between dimensionality and range further, we will determine the values of the parameter σ, the upper critical range σ_u and the lower critical range σ_ℓ. For $\sigma < \sigma_u$ the critical behaviour of the system is classical, while for $\sigma > \sigma_\ell$ no symmetry breaking takes place. We will then calculate the critical behaviour in an expansion in $\sigma - \sigma_u$ and $\sigma_\ell - \sigma$. This thesis is divided into two main chapters. In chapter one we study a percolation model defined on a one dimensional chain. The set of all pairs of points constitute the set of bonds. The bond occupation probability depends on the bond length as

$$P(i,j) = \left[1 - e^{-\frac{|K|}{|i-j|^{\sigma}}}\right]$$

Section one is an introduction to the model. In section two we review the basic scaling ideas, as applied to percolation. Singularities near the onset of percolation are connected to the divergence of the cluster size. In section three we show that our model is equivalent to an $s \to 1$ limit of an s-state Potts model with long range interactions. Section four contains one of the main results of this thesis. We generalize Anderson-Yuval-Cardy ideas to construct a renormalization group which is exact near the lower critical range of the models here considered. In section five we apply this renormalization group to the critical behaviour of percolating clusters. We conclude this chapter with a summary and some general remarks, in section six.

In chapter three we study a spin glass model described by the hamiltonian

$$H = \sum_{ij} \frac{J_{ij} S_i S_j}{|i-j|^\sigma}$$

where the J_{ij} are independent random variables with zero mean and finite variance. Section one is an introduction to the model. In section two we exhibit the spin glass instability by means of a diagrammatic high temperature expansion. Section three contains the derivation of the continuum field theory which is equivalent to our model in the limit of very long wavelengths. This field theory is analysed using Wilson recursion relations in section four. In section five we apply the general renormalization group equations derived in chapter one, to analyse the critical behaviour of the spin glass near its lower critical range. We conclude this chapter with a discussion and a summary of the results obtained.

Chapter II

ONE DIMENSIONAL PERCOLATION MODEL

2.1 INTRODUCTION TO THE MODEL

In a bond percolation[1] problem one is given a set of sites connected by bonds. A bond b can be independently occupied (closed) or empty (open) with probability p(b) and 1-p(b), respectively. Two sites belong to the same cluster if there is a chain of occupied bonds connecting them.

There is a critical probability p_c being the largest value of p for which a given site belongs to a cluster of finite size with probability one. For $p \geq p_c$ there is a non zero probability that a given occupied site belongs to an unbounded cluster. This probability is known as the percolation probability $P_\infty(p)$.

We will consider sites on a one dimensional chain. The set of bonds is the set of pairs $\{(i,j)\}$. A bond (i,j) is occupied with probability

$$p(i,j) = 1 - e^{\frac{K}{|i-j|^\sigma}} \qquad 1.1$$

For large separations this is asymptotically

$$p(i,j) \simeq \frac{-K}{|i-j|^\sigma} \qquad 1.2$$

K , a negative constant so that $0 < p(i,j) < 1$, and $\sigma > 0$, a positive constant controlling the decay of $p(i,j)$, are the parameters characterizing the statistical distribution of bonds.

This problem has several applications. It gives the zero temperature limit of a dilute ferromagnet with long range interactions. Bonds between distant spins are less likely to be occupied than bonds between near neighbours and eq. 1.1 models a power law decay of the bond dilution probability. Above the percolation threshold the system has a net magnetization that vanishes continuously as $K \to K_c$.

Another practical application is to a communication network running along a one dimensional structure. All users are directly connected, but long lines are more likely to malfunction than short ones. Below the percolation threshold each user can communicate with a finite number of users only. Above the percolation threshold there exists a finite probability that a given user is connected to an infinite number of other users. Note that two users, a and b, can communicate even if the direct line between them is open provided they can find a chain of intermediate lines connecting them.

Interconnected sites are called clusters. A cluster containing s sites will be referred to as an s-cluster. The number of such clusters for a finite chain grows linearly with the length of the chain. We will divide the number of clusters by the length of the chain. The average value of this ratio in the thermodynamic limit is denoted by n_s . It is a function of the parameters σ and K that define the probability distribution used to perform the average.

Percolation occurs when an infinite cluster of sites is formed with probability one (in the limit of an infinite chain).

The main questions we would like to address are two: 1) For which values of σ is there a percolation threshold. 2) When a percolation threshold exists, different cluster statistics acquire a singular part which we would like to calculate.

The typical cluster radius, ξ, diverges as $K \to K_c$. The existence of a divergent length scale produces nonanalytic behaviour in the cluster properties. This is reminiscent of second order phase transition and this analogy will be made more precise in the next sections. It is customary to introduce critical exponents to describe the leading singularities.

$$\left[\sum_s n_s(K) \right]_{sing} \simeq |K - K_c|^{2-\alpha} \qquad 1.3$$

$$\left[\sum_s s\, n_s(K) \right]_{sing} \simeq |K - K_c|^{\beta} \qquad 1.4$$

$$\left[\sum_s s^2 n_s(K) \right]_{sing} \simeq |K - K_c|^{-\gamma} \qquad 1.5$$

$$\left[\sum_s s\, n_s(K)\, e^{-hs} \right]_{sing} \simeq h^{\frac{1}{\delta}} \qquad 1.6$$

$$\xi(K) \simeq \frac{1}{|K - K_c|^{\nu}} \qquad 1.7$$

The subscript sing denotes the leading nonanalytic part of the subscripted quantity. For example, if for values of K close to K_c a quantity f(K) is given by

$$f = f_0 + (K-K_c)f_1 + |K-K_c|^{1.5} f_2 + |K-K_c|^{1.7} f_3 + (K-K_c)^2 f_4 \qquad 1.8$$

then

$$f_0 + f_1(K-K_c) + f_2(K-K_c)^2 \qquad 1.9$$

is referred to as the analytic backround. $[f]_{sing} = f_2 |K-K_c|^{1.5}$ and $f_3 |K-K_c|^{1.7}$ is a correction to scaling.

It is convenient to define a pair connectedness correlation function $C(x,y)$.

$C(x,y)$= Probability that x and y belong to the same cluster 1.10

The sum of the pair connectedness function over all sites is easily related to the mean cluster size. Denoting expectation values by E, and the charcteristic function of an event B by χ_B we find

$$\sum_{x'} C(x,x') = \sum_{x'} E(\chi_{\{x,x' \text{ belong to the same cluster}\}}) = E\binom{\text{size of cluster}}{\text{containing } x} \qquad 1.11$$

Eq. 1.5 can be written as

$$\sum_{x} C(0,x,K) \simeq \frac{1}{|K-K_c|^{\gamma}} \qquad 1.12$$

While at $K=K_c$

$$C(x,x'\ K_c) \simeq \frac{1}{|x-x'|^{d-2+\eta}} \qquad 1.13$$

defines the exponent η.

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2.2 SCALING THEORY OF PERCOLATION CLUSTERS

Scaling concepts developed for understanding critical phenomena can be successfuly used to describe geometric problems like percolation.[1] The bond probability distribution is a function of two variables K and σ. In this section we take a fixed value of σ in the range $1<\sigma<2$. For a given σ in this range there exists a critical value of K, $K_c(\sigma)$, at which percolation occurs. We will prove these assumptions in section 2.5. In this section we will discuss phenomenologically the singular behaviour of the cluster statistics near K_c. We begin the section with brief explanation of the scaling assumption and its consequences as applied to percolation theory. We then define the critical exponents to be calculated in section 2.5. The cluster numbers were defined in section 2.1 as

$$n_s(K) = \frac{\text{average number of clusters of size } s}{\text{number of lattice sites}} \qquad 2.1$$

The scaling assumption states that the critical behaviour near percolation is dominated by clusters of size

$$\xi^d \simeq \frac{}{|K-K_c|^{\nu d}} \qquad 2.2$$

d is the space dimensionality. This defines an exponent ν. At the percolation threshold

$$n_s(K_c) \simeq \frac{1}{s^{\tau}} \qquad 2.3$$

defining the exponent τ. The scaling assumption states that all the singularities near the percolation threshold arise from the divergence of ξ as $K \to K_c$. In other words

$$n_s(K) = n_s(K_c)\, f\left(\frac{s}{\xi(K)}\right) \qquad 2.4$$

f being some regular function. Since $n_s(K_c) \simeq \frac{1}{s^\tau}$ it follows that

$$n_s(K) \simeq \frac{1}{s^\tau} f\left(\frac{s}{\xi(K)}\right) \qquad 2.5$$

The main consequence of the scaling assumption is that we can express all the singularities related to the onset of percolation in terms of only two exponents ν and τ. The critical exponents are defined by:

$$\sum_s n_s(K) = \text{total number of cluster /lattice size} \simeq |K_c - K|^{2-\alpha} \qquad 2.6$$

$$P_\infty(K) = \text{probability of having the origin in the infinite cluster} \simeq |K - K_c|^\beta \qquad 2.7$$

$$S(K) \equiv \text{average cluster size} \simeq \sum_s n_s(K)\, s^2 \simeq \frac{}{|K - K_c|^\gamma} \qquad 2.8$$

At the percolation threshold

$$\sum_s n_s(K_c)\, s\, e^{-hs} \simeq h^{\frac{1}{\delta}} \qquad 2.9$$

The pair connectedness function c(x,y) defined by the probability for x and y to belong to the same cluster, at K_c, decays as

$$C(x, x'\, K_c) \simeq \frac{}{|x - x'|^{d-2+\eta}} \qquad 2.10$$

In the critical region $C(x,0,K)$ assumes the scaling form

$$C(x, 0, K) \simeq \frac{1}{|x|^{d-2+\eta}} f\left(\frac{x}{\xi}\right) \qquad 2.11$$

Now, we relate the cluster exponents ν and τ to the other critical exponents α, β, γ, δ, and η. In order to evaluate the singular part of any moment $\left[\sum_s n_s s^k\right]_{sing}$, we substitute the scaling form of the cluster distribution (eq. 2.5) and we replace in the sum each factor of s by the typical size $s \simeq s_\xi \simeq \xi^d$. The summation over all cluster sizes gives an additional factor of $s_\xi = \xi^d$. Finally we relate ξ and $K-K_c$ via eq. 2.2 . Hence, from the scaling assumption it follows that

$$\sum_s n_s(K) \simeq \xi\, n_\xi(K) \simeq \frac{\xi}{\xi^\tau} f(1) = \xi^{1-\tau} = (K-K_c)^{d\nu(\tau-1)} \qquad 2.12$$

$$\therefore \quad 2-\alpha = (\tau-1)\, d\nu \qquad 2.13$$

$$\sum_s n_s(K)\, s \simeq \xi^2 n_\xi(K) = \xi^{2-\tau} \simeq |K-K_c|^{d\nu(2-\tau)} \qquad 2.14$$

$$\therefore \quad \beta = (2-\tau)\, d\nu \qquad 2.15$$

$$\sum_s n_s(K)\, s^2 \simeq \xi^3 n_\xi(K) \simeq \xi^{3-\tau} \simeq \frac{1}{|K-K_c|^{(3-\tau)d\nu}} \qquad 2.16$$

$$\therefore \quad \gamma = (3-\tau)\, d\nu \qquad 2.17$$

$$\sum_s n_s(K_c)\, e^{-hs} \simeq s^2\, n_s(K_c)\Big|_{s=\frac{1}{h}} \simeq \frac{1}{h^{2-\tau}} \qquad 2.18$$

$$\therefore \quad \delta = \tau - 2 \qquad 2.19$$

$$C(0, x, K_c) = \sum_{s \geq |x|^d} s\, n_s(K_c) \simeq \sum_{s \geq |x|^d} \frac{s}{s^\tau} \simeq \frac{1}{|x|^{(\tau-2)d}} \qquad 2.20$$

$$\therefore \quad d-2+\eta = (\tau-2)\, d \qquad 2.21$$

thus relating all the critical exponents to ν and τ .

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2.3 MAPPING ONTO A ONE STATE POTTS MODEL

The s state Potts model[1] is a generalization of the Ising model. It is important for the study of critical properties of quenched random systems. The zero state limit describes the statistics of trees on a lattice[2] while the one state limit is related to the critical behaviour of percolating clusters.[3]

This mapping of a noninteracting random system onto an interacting translationally invariant model is important because it allow us to use the machinery developed for analysing critical phenomena in spin systems. It is also illuminating to see the correspondence between the critical behaviour of certain limit of a spin model and a purely geometrical phenomenon like percolation.

In this section we show, following the line of argument of ref. 3 that our one dimensional percolation problem can be mapped onto a one dimensional s=1 state Potts model with long range interactions. The partition function of the Potts model for general s is given by

$$Z = \mathrm{tr}\, e^{-\beta H}$$

$$-\beta H = \sum_{i<j} \frac{s|K|\left[\delta^{\sigma(i)\sigma(j)} - 1\right]}{|i-j|^{\sigma}} + h\sum_{i}\left[s\,\delta^{\sigma(i)1} - 1\right] + hN(1-s) \qquad 3.1$$

Here, σ is a parameter characterising the range of the interaction. We use the identity

$$e^{\frac{s|K|\left[\delta^{\sigma(i)\sigma(j)} - 1\right]}{|i-j|^{\sigma}}} = e^{-\frac{|K|s}{|i-j|^{\sigma}}}\left\{1 + \delta_{\sigma(i)\sigma(j)}\left[e^{\frac{|K|s}{|i-j|^{\sigma}}} - 1\right]\right\} \qquad 3.2$$

to perform a high temperature expansion

$$Z(s, K, h) = \mathrm{tr}\prod_{(ij)} e^{-\frac{|K|s}{|i-j|^{\sigma}}}\left\{1 + \delta_{\sigma(i)\sigma(j)}\left[e^{\frac{|K|s}{|i-j|^{\sigma}}} - 1\right]\right\} e^{hs\sum_i\left[\delta^{1\sigma(i)} - 1\right]} \qquad 3.3$$

Expanding the product we obtain a sum over all possible bond configurations. An occupied bond is associated with a factor $\left[1 - e^{-\frac{|K|s}{|i-j|^\sigma}}\right]\delta_{\sigma(i)\sigma(j)}$ while an empty bond carries a factor $e^{-\frac{|K|s}{|i-j|^\sigma}}$. The trace factorizes into a product of connected clusters

$$Z(s,K,h) = \sum_g \prod_{(ij)\in g} \left(1 - e^{-\frac{|K|s}{|i-j|^\sigma}}\right)\delta_{\sigma(i)\sigma(j)} \prod_{(ij)\notin g} e^{-\frac{|K|s}{|i-j|^\sigma}} \prod_j e^{sh(\delta^{\sigma(i)1}-1)}$$

$$= \sum_g \prod_{(ij)\notin g} e^{-\frac{|K|s}{|i-j|^\sigma}} \left[1+(s-1)e^{-sh}\right] \prod_k \left[1+(s-1)e^{-hsk}\right]^{n_k(g)} \prod_{(ij)\in g}\left(1 - e^{-\frac{|K|s}{|i-j|^\sigma}}\right) \qquad 3.4$$

$n_k(g)$ denotes the number of connected clusters of exactly k sites on the graph g . Defining isolated sites as clusters of unit size we get

$$-f = \lim_{s\to 1}\lim_{N\to\infty} \frac{\ln Z}{N(s-1)} = -h + \lim \frac{\ln\left\{\sum_g w(g) \prod_k \left[1+(s-1)e^{-hk}\right]^{n_k(g)}\right\}}{(s-1)N} \qquad 3.5$$

$$\simeq -h + \frac{1}{N} \ln \sum_g w(g) \sum_k n_k(g)\, e^{-kh} \qquad 3.6$$

The weights w(g) in eq. 3.6

$$w(g) = \prod_{(ij)\in g}\left(1 - e^{-\frac{|K|}{|i-j|^\sigma}}\right) \prod_{(ij)\notin g} e^{-\frac{|K|}{|i-j|^\sigma}} \qquad 3.7$$

are exactly the weights of the one dimensional percolation problem. We have shown that the $s \to 1$ limit of the free energy of a Potts model with long range interactions is the generating functional of the cluster statistics of our one dimensional percolation problem.

$$-f = \sum_k \frac{\langle n_k(g)\rangle}{N} e^{-kh} - h = \sum_k n_k\, e^{-kh} - h \qquad 3.8$$

The magnetic field h has a nice physical interpretation in the percolation problem. We modify the original problem by adding one additional "ghost site" outside the chain (see figure 1) and connecting bonds from the ghost site to each site on the chain. These new bonds are occupied with probability $1-e^{-h}$ (and are empty with probability e^{-h})

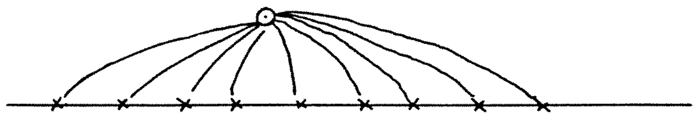

Figure 1: Modified percolation problem: the chain and its ghost site

We now have an infinite cluster even if $p < p_c$, provided $h > 0$ (all the sites connected to the ghost site constitute an infinite cluster). This is precisely the effect of turning on a magnetic field in a spin system, thereby creating a finite magnetization at all temperatures.

The cluster numbers $\langle n_s(K)\rangle_h$ in the presence of the ghost site are easily related to the cluster numbers in the original problem $\langle n_s\rangle_{h=0}$. For a cluster to contain s sites it must have the same structure as before, and in addition must have all its connections to the ghost site open. Therefore

$$\langle n_s\rangle_h = \langle n_s\rangle_{h=0}\, e^{-hs} \qquad 3.9$$

Eq. 3.8 can be interpreted as the generating functional of the cluster statistics in the presence of the additional ghost site. In the limit $s \to 1$ the magnetization $-\frac{\partial f}{\partial h}$ corresponds to $-P_\infty$ where P_∞ is the probability that a given site belongs to the infinite cluster.

The susceptibility corresponds to

$$-\frac{\partial^2 f}{\partial h^2} = \sum_K n_K K^2 = \text{average cluster size} \equiv S(K) \qquad 3.10$$

The percolation threshold is signaled by a nonvanishing .

$$P_\infty(K) \simeq |K - K_c|^\beta \qquad 3.11$$

The divergence of the magnetic susceptibility corresponds the divergence of the average cluster size.

$$S(K) \simeq \frac{1}{(K-K_c)^\gamma} \qquad 3.12$$

The anomalous dependence of the magnetization on the magnetic field at criticality corresponds to

$$\sum_K n_K K e^{-Kh} \Big|_{K=K_c} \simeq h^{\frac{1}{\delta}} \qquad 3.13$$

In a magnetic system the susceptibility diverges when spin-spin correlations become long ranged. In the percolation problem the mean cluster size diverges because of the slow decay of the pair connectedness function. We can define

$$C(X) = \text{Prob}\ (0, x \text{ belong to the same cluster}) - P_\infty^2 \qquad 3.14$$

in terms of which we can write S(K):

$$\sum_x C(x, K) = S(K)$$

We can relate C(x) to an $s \rightarrow 1$ limit of a Potts spin-spin correlation function.

$$\langle [s\,\delta^{\sigma(x)\sigma(x')} - 1] \rangle = \frac{1}{Z} \operatorname{tr} \prod_{(ij)} e^{\frac{-|K|s}{|i-j|^\sigma}} [1 + \delta_{\sigma(i)\sigma(j)} (e^{\frac{s|K|}{|i-j|^\sigma}} - 1)] [s\,\delta_{\sigma(x)\sigma(x')} - 1]$$

$$= \sum_g \operatorname{tr} \prod_{(ij)\in g} (1 - e^{-\frac{|K|s}{|i-j|^\sigma}}) \delta_{\sigma(i)\sigma(j)} \prod_{(ij)\notin g} e^{-\frac{|K|s}{|i-j|^\sigma}} [s\,\delta_{\sigma(x)\sigma(x')} - 1] \Big/ Z \qquad 3.16$$

$$\frac{1}{Z} \sum_{\substack{g \\ g \text{ having } x, x' \text{ in the} \\ \text{same cluster}}} w(g)\,(s-1) + \frac{1}{Z} \sum_{\substack{g \\ \text{having } x, x' \text{ in} \\ \text{different clusters}}} \prod e^{-\frac{|K| s}{|i-j|^{\sigma}}} \operatorname{tr}\left[s\,\delta_{\sigma(x)\,\sigma(x')} - 1 \right]$$

$$= (s-1) \ \text{PROB} \{ x, x' \text{ belong to the same cluster} \}$$

Using eq. 3.14 and eq. 3.16 we conclude

$$\lim_{s \to 1} \frac{1}{(s-1)} \left\langle s\,\delta^{\sigma(x)\,\sigma(x')} - 1 \right\rangle = C(x, x') \qquad 3.17$$

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2.4 DISCRETE CHAIN WITH LONG RANGE INTERACTIONS

The main topic of this thesis is one dimensional random systems with long range interactions. In this thesis we show that the one dimensional percolation problem (via the Kasteleyn - Fortuin trick) and the one dimensional spin glass problem (via the replica trick) can be mapped onto a chain of spins with long range interactions. The spins can be in a finite discrete number of states.

The critical behaviour of discrete one dimensional models with inverse power interactions is of considerable theoretical interest. Ising models with $\frac{1}{r^\sigma}$ interactions are known rigorously to exhibit long range order if $1 < \sigma < 2$ and have no transition if $\sigma > 2$.[1] The $\sigma = 2$ Ising model is of interest because it can be mapped onto the Kondo problem. The one dimensional row represents imaginary time in the path integral formalism, the Ising chain represents the time history of the impurity.

Within this framework Anderson and Yuval[2] derived Renormalization group equations (those equations were rediscovered by Kosterlitz and Thouless three years later in connection with the XY model).[3] They rewrote the spin partition function as a partition function of gas of interacting defects. The renormalization group equations were obtained by integrating out tightly bound pairs of defects. Later on Cardy[4] generalized Anderson -Yuval ideas to treat the general discrete model interacting via $\frac{1}{r^2}$ interactions. Surprisingly the Anderson Yuval technique used to derive the renormalization group equations (identical in spirit to Wilson's decimation procedure) works for this general case as well. The general discrete chain is interesting in view of its relationship to higher spin generalizations of the Kondo problem. It has also some features which resemble two dimensional spin models with continuous symmetries and their four dimensional gauge theory counterpart.

In this section we will generalize and extend Cardy's work in two directions. First we will consider $\frac{1}{r^{2-\varepsilon}}$ interactions. This will be the first step towards a systematic ε- expansion around the lower critical range. This expansion is similar to the Nelson-Fisher[5] extension of the XY model and to the Kosterlitz[6] extension of the vector spin chain. Second, we will include a generalized magnetic field term in the hamiltonian. This will be useful in the computation of critical exponents.

We consider a one dimensional chain, with periodic boundary conditions. The sites are labeled by integers, at each site i there is a spin variable that can be in one of s possible states. We denote the possible states by S_α $\alpha=1\cdots s$; sometimes we will denote the state S_α simply by its index α . The hamiltonian is defined by

$$-\beta H = \sum_{m<n} \frac{K(\sigma_m,\sigma_n)}{(a|m-n|)^\sigma} + \sum_n h(\sigma_n) \qquad 4.1$$

a is the lattice spacing, K is a ferromagnetic interaction i.e.

$$K(\alpha,\beta) = K(\beta,\alpha) < K(\alpha,\alpha)$$

h is a function of the spin variable state. If σ is an Ising spin the states are either + or − and $h(+) = h$, $h(-) = -h$ is the usual magnetic field term. A function of the state of the spin variable $h(x)$ is a suitable generalization of the magnetic field term for the general discrete model. We will impose the restriction $\sum_{\alpha=1}^{S} h(s_\alpha) = 0$ 4.2

which preserves the zero of the ground state energy. We will use the notation

$$V(n) = \begin{cases} \dfrac{1}{|n|^\sigma} & n \neq 0 \\ 0 & n = 0 \end{cases} \tag{4.3}$$

We integrate the action by parts twice . We define

$$H(r) = \sum_{s=0}^{r} V(s) + C_1$$

$$U(n) = \sum_{r=0}^{n-1} H(r) + C_2 \tag{4.4}$$

so that

$$H(r) - H(r-1) = V(r) \qquad U(n+1) - U(n) = H(n)$$

then

$$\sum_{m<n} V(n-m) K(\sigma_m \sigma_n) = \sum_{m<n} [H(n-m) - H(n-m-1)] K(\sigma_m \sigma_n)$$

$$= \sum_{m<n} H(n-m)[K(\sigma_m \sigma_n) - K(\sigma_m \sigma_{n+1})] - H(0) \sum_n K(\sigma_n \sigma_{n+1})$$

$$= \sum_{m<n} U(n-m)\left[K(\sigma_m\sigma_{n+1}) + K(\sigma_{m+1}\sigma_n) - K(\sigma_m\sigma_n) - K(\sigma_{m+1}\sigma_{n+1})\right]$$

$$+ U(o)\sum_n K(\sigma_n\sigma_{n+1})$$

We now adjust the constants C_1 , C_2 in eq. 4.4 so that $U(n)\to\infty$ as $n\to\infty$ for $\varepsilon>0$, and $U(n)\simeq \log|n|$ for $\varepsilon=0$, without a constant term. With this choice $U(o)>0$, and

$$U(n) = \frac{1}{(2-\sigma)(1-\sigma)}\left[|n|^{2-\sigma} - 1\right] \qquad 4.6$$

The action

$$\sum_{m<n}\left[K(\sigma_m\ \sigma_{n+1}) + K(\sigma_{m+1}\ \sigma_n) - K(\sigma_m\ \sigma_n) - K(\sigma_{m+1},\sigma_{n+1})\right]U(n-m) \qquad 4.7$$

describes the interaction between defects. We say there is a defect of type $\alpha\beta$ at site i if $S_i = S_\alpha,\ S_{i+1} = S_\beta\ \alpha\neq\beta$. Note that if $\sigma_m = \sigma_{m+1}$ (or if $\sigma_n = \sigma_{n+1}$) i.e. if there is no defect present at site m (n)

$$K(\sigma_m\ \sigma_{n+1}) + K(\sigma_{m+1}\ \sigma_n) - K(\sigma_m\ \sigma_n) - K(\sigma_{m+1}\sigma_{n+1}) = 0 \qquad 4.8$$

so 4.7 indeed describes an interaction between localized defects. We can think of

$$y_{\alpha\beta} = e^{U(o)\,K(\alpha,\beta)} \qquad 4.9$$

as a fugacity for creating an α,β defect.

Using eqs. 4.5 - 4.8 we rewrite the partition function as

$$Z = \sum_{n=0}^{\infty}\ \sum_{\alpha_1\cdots\alpha_n} y_{\alpha_1\alpha_2}\cdots y_{\alpha_n\alpha_1}\int\frac{dr_1}{a}\cdots\frac{dr_n}{a}\prod_{j=1}^{n}\theta(r_{j+1}-r_j-a)\, e^{G_n[r_1\cdots r_n]} \qquad 4.10$$

$$\mathcal{A}[r_1 \cdots r_n] = \sum_{i<j} \frac{\left[\left|\frac{r_i - r_j}{a}\right|^{2-\sigma} - 1\right]}{(2-\sigma)(\sigma-1)} \left[K(\alpha_i \alpha_j) + K(\alpha_{i+1}\alpha_{j+1}) - K(\alpha_{i+1}\alpha_j) - K(\alpha_i \alpha_{j+1})\right]$$

$$+ \sum_i h(\alpha_i)\left(\frac{r_i - r_{i-1}}{a}\right) \qquad 4.11$$

We impose periodic boundary conditions $r_{n+1} = r_n$. Originally the coordinates r_i, describing the position of the defects took values ia, with $i \in \mathbb{Z}$. In eq. 4.10 we replaced the original system of kinks on the lattice by a continuous one, the function $\theta(r_{i+1} - r_i - a)$ provides a cut-off of size a. Clearly, the critical behaviour of the system is insensitive to the choice of regularization procedure. The renormalization group equations are derived from the fact that the physics is independent of the particular value of the microscopic length, a, defined by the lattice spacing. A change in a is equivalent to a redefinition of the fugacities $y_{\alpha\beta}$, coupling constants $K_{\alpha\beta}$, generalized magnetic field h_α and ground state energy.

In other words, changing the cut-off $a \to a + \delta a$ can be compensated by changing $k \to k + \delta k$, $y \to y + \delta y$, $h \to h + \delta h$ and adding a constant to the free energy so as to get a partition function of the form 4.10 with y,a,k,h, replaced by $y + \delta y$, $a + \delta a$, $K + \delta K$, $h + \delta h$.

$$Z(a \; y_{\alpha\beta} \; K_{\alpha\beta} \; h_\alpha) = \exp \Delta E(\ell) \; Z[e^{\ell} a \;\; y_{\alpha\beta}(\ell) \; K_{\alpha\beta}(\ell) \;\; h_\alpha(\ell)]$$

To verify this assertion and to determine the equations governing the flow of y(l), k(l), h(l) we change the cut-off $a \to a e^{\ell}$ in eq. 4.10. The change in Z due to the change in the fundamental length appearing explicitly in eq. 4.10 is

$$\frac{1}{a^n} \exp\left\{ \sum_{i<j} \frac{\left[\left(\frac{\pi_i - \pi_j}{a}\right)^{2-\sigma} - 1\right]}{(2-\sigma)(\sigma-1)} \left[K(\alpha_i \alpha_j) + K(\alpha_{i+1} \alpha_{j+1}) - K(\alpha_{i+1} \alpha_j) - K(\alpha_i \alpha_{j+1})\right] + \sum_i h(\alpha_i)(\pi_i - \pi_{i-1}) \right\} \to \frac{1}{(e^{\ell} a)^n} \exp\left\{ \sum_{i<j} \frac{\left[\left(\frac{\pi_i - \pi_j}{a e^{\ell}}\right)^{2-\sigma} - 1\right]}{(2-\sigma)(\sigma-1)} \left[K(\alpha_i \alpha_j) + K(\alpha_{i+1} \alpha_{j+1}) - K(\alpha_i \alpha_{j+1}) - K(\alpha_{i+1} \alpha_j)\right] + \sum_i h(\alpha_i) \frac{(\pi_i - \pi_{i-1})}{a e^{\ell}} \right\} \qquad 4.12$$

This can be compensated by changing

$$K(\alpha,\beta) \to e^{\ell(2-\sigma)} K(\alpha,\beta) \sim K(\alpha,\beta) + \ell(2-\sigma) K(\alpha,\beta) \qquad 4.13$$

$$h(\alpha) \to e^{\ell} h(\alpha) \sim h(\alpha) + \ell h(\alpha) \qquad 4.14$$

$$\prod_i y(\alpha_i \alpha_{i+1}) \to \prod_i e^{\ell} y(\alpha_i \alpha_{i+1})\, e^{\sum_{i<j} \frac{\ell(2-\sigma)}{(2-\sigma)(\sigma-1)} [K(\alpha_i \alpha_j) + K(\alpha_{i+1} \alpha_{j+1}) - K(\alpha_{i+1} \alpha_j) - K(\alpha_i \alpha_{j+1})]} \qquad 4.15$$

Using the identity

$$\frac{1}{2} \sum_{i \neq j} \left[K(\alpha_{i+1}, \alpha_j) + K(\alpha_i \alpha_{j+1}) - K(\alpha_i \alpha_j) - K(\alpha_{i+1} \alpha_{j+1}) \right]$$

$$= \sum_{i \neq j} \left[K(\alpha_{i+1} \alpha_j) - K(\alpha_i \alpha_j) \right] = - \sum_i K(\alpha_{i+1}, \alpha_i)$$

We rewrite eq. 4.15 as

$$y(\alpha,\beta) \rightarrow y(\alpha,\beta)\, e^{\ell}\, e^{\frac{\ell K(\alpha,\beta)}{\sigma-1}} \sim y(\alpha,\beta) + \ell\left[1 + \frac{K(\alpha,\beta)}{\sigma-1}\right] \qquad 4.16$$

The second effect of changing the cut-off is to change

$$\theta(\pi_{j+1} - \pi_j - a) \rightarrow \theta(\pi_{j+1} - \pi_j - a e^{\ell}) \doteq \theta(\pi_{j+1} - \pi_j - a) - a\,\delta(\pi_{j+1} - \pi_j - a) \qquad 4.17$$

The delta function juxtaposes each neighbouring kink pair in turn. Physically this corresponds to integrating out pairs of tightly bound pairs of kinks which renormalize the interaction between widely separated kinks. Consider the factor involving two kinks i, i+1 and a third kink at j. As we juxtapose i and i+1 we generate (see fig. 2)

$$\Delta Z = -\ell \sum_{n=0}^{\infty} \sum_{\alpha_1 \cdots \alpha_n} \sum_{i=1}^{n} \int \frac{d\pi_1 \cdots \widehat{d\pi_{i+1}} \cdots d\pi_n}{a^n}\, y_{\alpha_1\alpha_2} \cdots y_{\alpha_i \alpha_{i+1}}\, y_{\alpha_{i+1}\alpha_{i+2}} \cdots e^{G_n[\pi_1 \cdots \pi_n]} \Big|_{\pi_{i+1} = \pi_i + a} \qquad 4.18$$

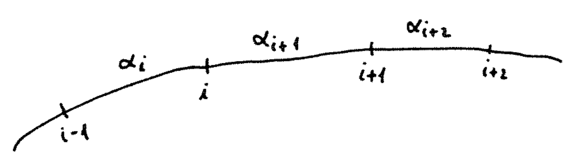

Figure 2: Kink description of spin configurations

Consider the term

$$e^{-\sum_{j>i+1} U(\pi_i - \pi_j)\left[K(\alpha_i\, \alpha_j) + K(\alpha_{i+1}\, \alpha_{j+1}) - K(\alpha_{i+1}\, \alpha_j) - K(\alpha_i\, \alpha_{j+1})\right]}$$

$$4.19$$

$$e^{-\sum_{j>i+1} U(\pi_i + a - \pi_j)\left[K(\alpha_{i+1}\, \alpha_j) + K(\alpha_{i+2}\, \alpha_{j+1}) - K(\alpha_{i+2}\, \alpha_j) - K(\alpha_{i+1}\, \alpha_{j+1})\right]}$$

in eq. 4.18 .When $\alpha_{i+2} \neq \alpha_i$ (see fig. 3) eq. 4.19 is equal to first order in $\frac{a}{|r_i - r_\gamma|}$ to

$$e^{-\sum_{\gamma>i} U(r_i - r_\gamma)\left[K(\alpha_i\,\alpha_\gamma) + K(\alpha_{i+2}\,\alpha_{\gamma+1}) - K(\alpha_i\,\alpha_{\gamma+1}) - K(\alpha_{i+2}\,\alpha_\gamma)\right]} \qquad 4.20$$

and the change 4.18 in the partition function can be absorbed in a renormalization of $y_{\alpha\beta}$

$$y_{\alpha\beta} \rightarrow y_{\alpha\beta} + \ell \sum_{\upsilon \neq \alpha,\beta} y_{\alpha\upsilon}\, y_{\upsilon\beta}\, e^{h(\upsilon)} \qquad 4.21$$

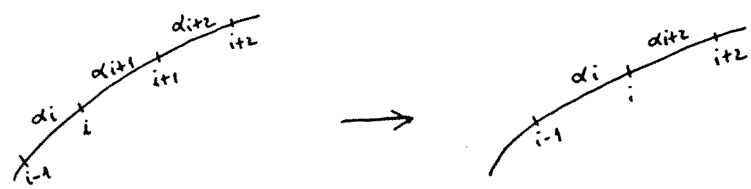

Figure 3: Configuration renormalizing the fugacity

When $\alpha_i = \alpha_{i+2}$ (see fig. 4) the leading term in eq. 4.19 vanishes and we need the next to leading term in $\frac{a}{|r_i - r_\gamma|}$, this is

$$\sum_{i=1}^{n} \exp \sum_{\gamma} \left[K(\alpha_{i+1}\,\alpha_\gamma) + K(\alpha_{i+2}\,\alpha_{\gamma+1}) - K(\alpha_{i+2}\,\alpha_\gamma) - K(\alpha_{i+1}\,\alpha_{\gamma+1})\right] \frac{1}{(1-\sigma)} \left|\frac{r_i - r_\gamma}{a}\right|^{1-\sigma} \qquad 4.22$$

Its contribution to the partition function is

$$\Delta Z = -\ell \sum_{k=1}^{n} y_{\alpha_1\alpha_2} \cdots y_{\alpha_k\alpha_{k+1}}\, y_{\alpha_{k+1}\alpha_{k+2}} \cdots \int \frac{dr_1}{a} \cdots \widehat{dr_k}\, \widehat{dr_{k+1}} \cdots \frac{dr_n}{a} \left\{ \right\} \qquad 4.23$$

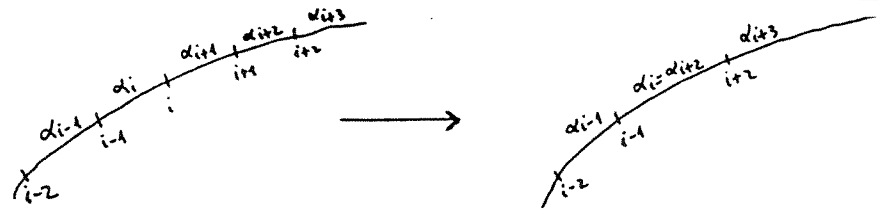

Figure 4: Configuration renormalizing the temperature

$$\{\,\} = e^{G_{n-2}[\pi_1 \cdots \pi_{k-1}\, \pi_{k+2} \cdots \pi_n] + h(\alpha_{k+1}) - h(\alpha_k)} \int_{\pi_{k-1}}^{\pi_{k+2}} \frac{d\pi_k}{a} \, \exp \frac{1}{(1-\sigma)} \sum_{j \neq k, k+1} \left| \frac{\pi_j - \pi_k}{a} \right|^{1-\sigma} [\,] \qquad 4.24$$

$$[\,] = K(\alpha_j\, \alpha_{k+1}) + K(\alpha_{j+1}\, \alpha_{k+2}) - K(\alpha_{j+1}\, \alpha_{k+1}) - K(\alpha_j\, \alpha_{k+2})$$

The main contirbution to the integral over $d\pi_k$ coming from the region $\pi_j - \pi_k >> a$ is:

$$\int_{\pi_{k-1}}^{\pi_{k+1}} \frac{d\pi_k}{a} \, e^{\frac{1}{(1-\sigma)} \sum_j \left| \frac{\pi_j - \pi_k}{a} \right|^{1-\sigma} \{ K(\alpha_j\, \alpha_{k+1}) + K(\alpha_{j+1}\, \alpha_{k+2}) - K(\alpha_{j+1}\, \alpha_{k+1}) - K(\alpha_j\, \alpha_{k+2}) \}} \qquad 4.25$$

$$\simeq \left(\frac{\pi_{k+2} - \pi_{k-1}}{a} \right) + \frac{\{ K(\alpha_j\, \alpha_{k+1}) + K(\alpha_{j+1}\, \alpha_{k+1}) - K(\alpha_{j+1}\, \alpha_{k+1}) - K(\alpha_j\, \alpha_{k+2}) \}}{(1-\sigma)\ (2-\sigma)} \left\{ \left(\frac{\pi_j - \pi_{k+2}}{a} \right)^{2-\sigma} - \left(\frac{\pi_j - \pi_{k-1}}{a} \right)^{2-\sigma} \right\}$$

The first term in eq. 4.25 generates a ground state energy renormalization and a renormalization of the magnetic field. The change in h necessary to compensate for this change is

$$h(\alpha_k) \to h(\alpha_k) + \ell \sum_{\alpha_{k+1}} [e^{h(\alpha_{k+1}) - h(\alpha_k)} - 1]\, y^2_{\alpha_k \alpha_{k+1}} \qquad 4.26$$

For infinitesimal fields this is

$$h(\alpha) \to h(\alpha) + \ell \sum_\beta y^2_{\alpha\beta}\, [h(\beta) - h(\alpha)] \qquad 4.27$$

Note that 4.27 preserves the condition

$$\sum_\alpha h(\alpha) = 0$$

Inserting the second term of eq. 4.25 into eq. 4.23 we find a change in the partition function that has to be compensated by a renormalization of K. The change in Z is:

$$\Delta Z = -\ell \sum_{k=1}^{n} y_{\alpha_1\alpha_2} \cdots y_{\alpha_{k-1}\alpha_k}\, y_{\alpha_k\alpha_{k+3}} \cdots \sum_{i\neq k,k+1,k+2} \int \frac{dr_1}{a} \cdots \widehat{dr_k}\, \frac{dr_{k+1}}{a} \cdots \frac{dr_n}{a}\, e^{G_{n-2}[r_1 \cdots r_{k-1} r_{k+2} \cdots r_n]} \times$$

$$\times \Big\{ \sum_{\upsilon} e^{h(\upsilon)-h(\alpha_k)} \Big\{ U(r_i - r_{k-1}) \left[K(\alpha_i,\upsilon) + K(\alpha_{i+1},\alpha_k) - K(\alpha_{i+1},\upsilon) - K(\alpha_i,\alpha_k)\right]\Big\} y^2_{\alpha_k\upsilon} \qquad 4.28'$$

$$+ \sum_{\upsilon} e^{h(\upsilon)-h(\alpha_{k-1})} \Big\{ U(r_i - r_{k-1}) \left[K(\alpha_{i+1},\upsilon) + K(\alpha_i,\alpha_{k-1}) - K(\alpha_{i+1},\alpha_{k-1}) - K(\alpha_i,\upsilon)\right]\Big\} y^2_{\alpha_{k-1}\upsilon}$$

Symmetrizing the double sum in the i, k variables we see this change can be absorbed in a renormalization of K.

$$K(\alpha,\beta) \longrightarrow K(\alpha,\beta) - \ell \sum_{\upsilon} y^2_{\alpha\upsilon} \left[K(\alpha,\beta) + K(\alpha,\upsilon) - K(\beta,\upsilon)\right] e^{[h(\upsilon)-h(\alpha)]}$$

$$- \ell \sum_{\upsilon} y^2_{\beta\upsilon} \left[K(\alpha,\beta) + K(\beta,\upsilon) - K(\alpha,\upsilon)\right] e^{[h(\upsilon)-h(\beta)]} \qquad 4.29$$

Note this renormalization preserves the condition $K(\alpha,\alpha)=0$.

Collecting eqs. 13, 14, 16, 21, 27, 29 we finally obtain the renormalization group equations which for infinitesimal magnetic fields read:

$$\frac{d}{d\ell} K(\alpha,\beta) = (2-\sigma)K(\alpha,\beta) - \sum_{\upsilon\neq\alpha} y^2_{\alpha\upsilon}\left[K(\alpha,\beta)+K(\alpha,\upsilon) - K(\beta,\upsilon)\right] - \sum_{\upsilon\neq\beta} y^2_{\beta\upsilon}\left[K(\alpha,\beta)+K(\beta,\upsilon) - K(\alpha,\upsilon)\right] \qquad 4.30$$

$$\frac{d}{d\ell} h(\alpha) = h(\alpha) + \sum_{\beta\neq\alpha} y^2_{\alpha\beta}\left[h(\beta) - h(\alpha)\right] \qquad 4.31$$

$$\frac{d}{d\ell} y(\alpha,\beta) = y(\alpha,\beta)\left[1 + \frac{K(\alpha,\beta)}{(\sigma-1)}\right] + \sum_{\upsilon\neq\alpha,\beta} y_{\alpha\upsilon}\, y_{\upsilon\beta} \qquad 4.32$$

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2.5 RENORMALIZATION GROUP EQUATIONS AND CRITICAL EXPONENTS

The critical behaviour near the percolation threshold, K ,is characterized by a typical cluster size which becomes very large. In section 2.2 we described a simple scaling picture of the transition. All the singular behaviour in the cluster statistics was related to the divergence of ξ as $K \to K_c$ and the decay of C(x,y) at $K=K_c$.

To provide a more quantitative description we need more sophisticated tools: the renormalization group and the ε-expansion.

In section 2.3 we established a connection between our one dimensional percolation problem, a geometric phase transition of non interacting (i.e independent) bonds and the $s \to 1$ limit of a strongly interacting ferromagnetic system exhibiting a thermal phase transition. This model is a particular case of the general discrete chain considered in section 2.4 . Indeed if

$$K(\alpha, \beta) = K \qquad \alpha \neq \beta \qquad 5.1$$

$$K(\alpha, \alpha) = 0 \qquad \alpha = 1 \cdots s \qquad 5.2$$

$$h(\alpha) = (s-1)h \quad \alpha = 1 \qquad h(\alpha) = -h \quad \alpha = 2 \cdots s \qquad 5.3$$

the hamiltonian 4.1 becomes, up to an additive constant, 3.1 describing the s-state Potts model with long range interactions. In the percolation problem the generalized magnetic field of section 2.4 becomes the ghost field of section 2.3. Note that $h(\alpha)$ as defined in eq. 5.3 satisfies the condition $\sum_\alpha h(\alpha) = 0$ (eq. 4.2). The relation 5.1 and the corresponding equation for the fugacity

$$y_{\alpha\beta} = y \qquad \alpha \neq \beta \qquad 5.4$$

are preserved by the renormalization group equations derived in section 2.4. We can immediately apply this formalism to our problem. Using eqs. 5.1 and 5.4 we find that eqs. 4.30 – 4.32 in the previous section reduce to

$$\frac{dK}{d\ell} = (2-\sigma)K - 2sKy^2 \qquad 5.5$$

$$\frac{dh}{d\ell} = h(1-sy^2) \qquad 5.6$$

$$\frac{dy}{d\ell} = y\left(1+\frac{K}{\sigma-1}\right) + (s-2)y^2 \qquad 5.7$$

This system of equations can have three different types of fixed points.

Consider first the fixed point at K=0. For finite s, our equations describe a magnetic system and K is proportional to the inverse temperature $K \propto -\frac{1}{T}$. A fixed point at K=0 is an infinite temperature fixed point describing a disordered phase. The renormalization group equations 5.5 - 5.7 are analytic in s. As $s \to 1$ K is related to the bond occupation probability (see eq. 3.7)

$$P(i,j) = 1 - e^{\frac{K}{|i-j|^\sigma}} \qquad 5.8$$

P(i,j) is the probability for having a bond of length $|i-j|$ occupied. A K=0 fixed point describes a situation where P=0, that is all the bonds are open and the system is completely disconnected.

The fixed point at $K=-\infty$, for finite s describes a magnetic system at zero temperature. This is a completely ordered phase. As $s \to 1$, again K is related via eq. 5.8 to the bond occupation probability. In the context of our percolation problem $K=-\infty$ is equivalent to P=1, so the $K=-\infty$ fixed point represent a state of perfect connectivity having an infinite cluster.

An infrared unstable fixed point at finite K describes, for integers $s \geqslant 2$, a thermal phase transition from a ferromagnetic to a paramagnetic phase. The value K_c at which the system is scale invariant corresponds to the critical temperature. When $s \to 1$ an infrared unstable fixed point K_c signals the existence of a percolation threshold.

$$P_c(i,j) = \left[1 - e^{\frac{K_c}{|i-j|^\sigma}}\right] \qquad 5.9$$

is the critical bond occupation probability. If the initial values of P are such that $|K|<|K_c|$ (i.e. if $P(i,j)$ falls off faster than $\frac{|K_c|}{|i-j|^\sigma}$) the system is disconnected while if $|K|>|K_c|$ an infinite cluster will form.

Inspecting eq. 5.5 we see that for $\sigma > 2$, and finite initial values of K we always flow towards the K=0 fixed point while for $\sigma < 2$ an infrared unstable fixed point exists. $\sigma=2$ is the lower critical range for one dimensional percolation with power law decay of the bond occupation probability.

Eqs. 5.5 - 5.7 are exact to first order in y and have an infrared unstable fixed point with $y^2 = O(2-\sigma)$. This fact allows us to calculate the critical exponents exactly, to first order in $\sqrt{\varepsilon}$, in a $\sqrt{\varepsilon}$- expansion. Throughout this section ε is defined as $\varepsilon = 2-\sigma$. From eqs. 5.5 - 5.7 we find a fixed point

$$K^* = -1 + \sqrt{\frac{\varepsilon}{2}} + O(\varepsilon) \qquad 5.11$$

$$y^{*2} = \frac{\varepsilon}{2} + O(\varepsilon^{3/2}) \qquad 5.12$$

Linearizing the renormalization group equations around the fixed point we find, in the limit $s \to 1$.

$$\begin{pmatrix} \frac{d}{dl}\delta K \\ \\ \frac{d}{dl}\delta y \end{pmatrix} = \begin{pmatrix} 0 & -4K^* y^* \\ \\ y^* & -y^* \end{pmatrix} \begin{pmatrix} \delta K \\ \\ \delta y \end{pmatrix} \qquad 5.13$$

$$\frac{dh}{dl} = h(1 - y^{*2}) \qquad 5.14$$

The linearized equations have one positive eigenvalue characterizing the divergence of the cluster size near threshold.

$$\xi \simeq \frac{1}{|K - K_c|^{\nu}}$$

$$\frac{1}{\nu} = \left(\frac{-1 + \sqrt{17}}{2}\right)\sqrt{\frac{\varepsilon}{2}} + O(\varepsilon) \qquad 5.15$$

From equation 5.14 , using the well known relation between the magnetic field eigenvalue and the exponent η we find

$$1 - y_*^2 = \left(\frac{d}{2} + 1 - \frac{\eta}{2}\right)$$

$$\eta = 1 + \varepsilon + O(\varepsilon^2) \qquad 5.16$$

As $\varepsilon \to 0$, $\nu \to \infty$ this is characteristic of systems near their lower critical dimensionality. Note that $\frac{1}{\nu}$ is not analytic in ε . This feature is common to other phase transitions in random systems.[1] Finally, from the relation of our model to the ferromagnetic system we can conclude that for $\sigma < 1$, the system is always connected. The existence of a percolation threshold is valid only for $0 < \varepsilon < 1$.

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2.6 SUMMARY

We presented a new percolation model. Bonds between points i and j on a one dimensional chain are occupied with probability

$$P(i,j) = 1 - e^{-\frac{|K|}{|i-j|^{\sigma}}} \qquad 6.1$$

The model has applications to one dimensional dilute ferromagnets with long range interactions and to one dimensional switching networks.

We studied the percolation transition as a function of the parameters σ and K, defining the bond probability distribution. For $\sigma > 2$ the system is always disconnected while for $\sigma < 1$ an infinite cluster always exists. For $1 < \sigma < 2$ the system has a percolation threshold at a certain critical value of K. To analyse the critical behaviour at threshold we mapped our percolation model onto a Potts like model with long range interactions. We derived renormalization group equations valid for near 2. We then calculated the divergence of the cluster size as we approach threshold and the decay of the pair connectedness function at threshold in a $\sqrt{\varepsilon}$ expansion. The singular behaviour of other cluster statistics can be derived using the scaling theory described in section 2.4.

Chapter III

ONE DIMENSIONAL SPIN GLASS MODEL

3.1 INTRODUCTION TO THE MODEL

In this chapter we will treat a one dimensonal spin chain with random long range interactions. Ths hamiltonian posesses the essential features of a spin glass. The purpose of this section is to present a brief overview of the status of the spin glass problem as a motivation for its study.

Spin glasses are magnetic substances with exchange interactions of random sign. The first materials exhibiting spin glass behaviour were obtained by diluting magnetic impurities in a noble metal matrix. Random exchange is caused the oscillating nature of the, conduction electron mediated, RKKY interaction between randomly located impurities. There are by now a wide range of materials including magnetic semiconductors and glasses that exhibit the same kind of magnetic behaviour. These materials exhibit a phase transition as they are cooled down below a critical temperature called the freezing temperature. At the transition the a.c. susceptibility exhibits a sharp cusp, and the non linear susceptibility diverges.[1] The specific heat is perfectly smooth at the transition.

Below the transition temperature, remanence hysteresis and slow relaxation phenomena are observed.

From the theoretical perspective, the essential ingredients necessary to produce spin glass behaviour are disorder and frustration,[2] a word coined to describe the effect of competing interactions. The frustration effect is responsible for the large ground state degeneracy which is characteristic of glassy substances.

The intuitive picture of the spin glass as a phase where all the spins are frozen in random directions is simple and appealing. However, the implementation of this idea within the framework of statistical mechanics has encountered considerable difficulties. One main obstacle is the lack of a natural order parameter that can be used to label the different ground states of the system.

Expectation values like $\overline{\langle S_i \rangle}$ vanish due to the randomness in the exchange. $\langle\ \rangle$ denotes thermal average while a bar denotes a configurational average over the random exchange.

The expectation value $\overline{\langle S_i \rangle^2}$ is not zero and its called the Edwards-Anderson[3] order parameter q. Unfortunately, q by itself is not sufficient to completely specify the spin glass phase. When it is calculated in the replica framework the natural identification $q = \langle S^\alpha S^\beta \rangle$ fails. To minimize the free energy one needs to assume $q^{\alpha\beta} = \langle S^\alpha S^\beta \rangle$ with a non trivial dependence on the replica indices.[4] In the limit when the number of replicas goes to zero the nxn matrix is parametrized by a function q(x). While max q(x)=q(1) is still identified with the Edwards-Anderson order parameter, the meaning of the other values of the function q(x) is not very transparent.[5] In particular the existence of a function as an order parameter implies the existence of a continuous symmetry that was not manifest in the original formulation of the problem.

A more intuitive approach to the problem was pioneered by H. Sompolinsky.[6]

He introduced relaxational dynamics and obtained a static limit of the dynamical problem by postulating a continuum of macroscopically long relaxation times. A startling consequence of his analysis is that any stable solution must violate the fluctuation dissipation theorem in the spin glass phase. He introduced an order parameter $\Delta(x)$ measuring the breakdown of linear response. The second order parameter is the value of the spin autocorrelation function $q(x)$ at a, macroscopically long, time scale x. Since the fluctuation dissipation theorem is obeyed in any finite system, it is not clear how it is lost in the thermodynamical limit. In fact the nature of the violation of the fluctuation dissipation theorem is still debated.[7]

Most of the theoretical work has been done on the Sherrington Kirkpatrick model:

$$H = \sum_{i=1}^{N} \sum_{j \neq i} J_{ij} S_i S_j \qquad 1.1$$

J_{ij} are gaussian random variables with

$$\overline{J_{ij}} = 0 \qquad \overline{J_{ij}^2} = \frac{J^2}{N}$$

In this model the thermodynamical limit and the range of the interactions is taken simultaneously to infinity. The thermodynamical limit is an extremely delicate mathematical operation and it is certainly questionable whether it can be interchanged with the infinite range limit. It is still not clear whether the peculiar features of this model manifest both in the statistical mechanical and in the dynamical treatment are artifacts of this particular limiting procedure or whether they are present in more realistic models as well. While most of the theoretical work[8] indicates that four is the lower critical dimensionality of the spin glass, there is by now ample experimental evidence for the existence of a sharp phase transition in three dimensional materials, and the lower critical dimensionality of the spin glass remains a major open problem.

In this chapter we will study a one dimensional model defined by the hamiltonian

$$H = \sum_{i \neq j} \frac{J_{ij} S_i S_j}{|i-j|^{\sigma}} \qquad 1.3$$

S_i are Ising spins and J_{ij} are independent random variables with probability distribution

$$\mathcal{P}(J_{ij}) = \frac{e^{-\frac{J_{ij}^2}{2J^2}}}{\sqrt{2\pi J^2}} \qquad 1.4$$

The model contains disorder in the couplings. It also has frustration. Even the pure antiferromagnet with $J_{ij}=-1$ is known to be frustrated. It has an infinite number of ground states with an arbitrary rational commensurability.[9] For $\sigma > 1/2$ the model defined by eqs. 1.3 - 1.4 is rigorously known to have a well defined thermodynamical limit.[10] Khanin and Sinai showed that except for a set of $\{J_{ij}\}$ of measure zero, the free energy per unit volume

$$\frac{1}{N} F(N, \beta, J_{ij}) = -\frac{1}{\beta N} \log \operatorname{tr}_S e^{-\beta \sum_{i \neq j} \frac{J_{ij} S_i S_j}{|i-j|^{\sigma}}} \qquad 1.5$$

has a well defined limit as $N \to \infty$ and this limit is independent of the particular values of the $\{J_{ij}\}$.

The parameter σ is a measure of the interaction range. In this one dimensional model it plays a role akin to dimensionality in short range models. When $\sigma < \frac{1}{2}$ the energy grows faster than the volume and fluctuations are suppressed. This is a limit of infinite dimensionality. When $\sigma > \frac{3}{2}$ the interaction is effectively short ranged and the system cannot exhibit symmetry breaking. This corresponds to a limit of low dimensionalities. We will study this model as a function of σ, and we will see it behaves very differently in different ranges or the parameter σ. This suggests that the three dimensional Edwards-Anderson model with short range interactions might have a very different behaviour from its infinite range, Sherrington-Kirkpatrick counterpart.

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3.2 THE SPIN GLASS INSTABILITY, HIGH TEMPERATURE EXPANSION

In this section we show that the model defined by eqs. 1.3 – 1.4 has a phase transition for sufficiently small σ by exhibiting a singularity in the high temperature expansion of the free energy and the non linear susceptibility.

The starting point of the high temperature expansion is

$$\overline{-\beta F} = \overline{\log \operatorname{tr} \prod_{i<j} \left[1 + \exp \frac{\beta J_{ij} S_i S_j}{|i-j|^\sigma} \right]} \qquad 2.1$$

The bar denotes the average over the bond variables J_{ij}. Using the identity

$$e^{S_i S_j a} = \operatorname{ch} a + \operatorname{sh} a \; S_i S_j = \operatorname{ch} a \left[1 + S_i S_j \operatorname{th} a \right]$$

valid for Ising spins, in eq. 2.1 we find

$$-\overline{\beta F} = \sum_{i<j} \overline{\log \operatorname{ch} \frac{\beta J_{ij}}{|i-j|^\sigma}} + N \log 2 + \overline{\log \frac{1}{2^N} \operatorname{tr} \prod_{i<j} \left[1 + S_i S_j \operatorname{th} \frac{\beta J_{ij}}{|i-j|^\sigma} \right]} \qquad 2.2$$

The first term in eq. 2.2 can be evaluated exactly and is given by

$$\frac{N}{2} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \sum_{j=1}^{\infty} \log \operatorname{ch} \frac{\beta J x}{|j|^\sigma} \frac{dx}{\sqrt{2\pi}} \simeq \frac{(\beta J)^2 N}{4} \sum_{j=1}^{\infty} \frac{1}{|j|^\sigma} \qquad 2.3$$

The second term can be represented in terms of diagrams. We label points by the integers $i \in Z$.

$$\frac{1}{2^N} \operatorname{tr} \prod_{i<j} \left[1 + S_i S_j \operatorname{th} \frac{\beta J_{ij}}{|i-j|^\sigma} \right]$$

is given by the sum over all different labelled graphs with an even number of lines at each vertex, and not more than one line joining each pair of points. To each line connecting points i and j we attach a factor $\operatorname{th} \frac{\beta J_{ij}}{|i-j|^\sigma}$ and with each vertex having 2n incoming lines we associate a factor $\frac{1}{2^{n-1}}$. Examples of these graphs are given in fig. 5
At this stage disconnected graphs like 1f appear in the expansion.

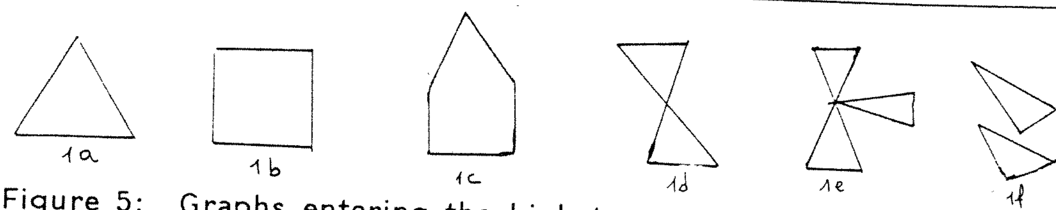

Figure 5: Graphs entering the high temperature expansion before configurational average

The next step is to take the logarithm in expression and perform the configurational average. Since $\overline{J_{ij}J_{\ell k}} = 0$ when $(ij) \neq (\ell k)$, only lines joining the same labelled vertices correlate with each other.

$$\log \overline{[1 + \triangle + \square + \cdots]} = \overline{[\triangle + \square + \cdots]} - \frac{1}{2}\overline{[\triangle + \square + \cdots]^2} + \frac{1}{3}\overline{[\triangle + \square + \cdots]^3} + \cdots \qquad 2.4$$

Eq. 2.4 can be represented as a sum over all labelled closed loop graphs. Since $\overline{(th\, \beta J_{ik})^{\ell}} = 0$ when k is odd, vertices are now connected by an even number of lines. Examples of graphs contributing to eq. 2.4 are given in fig. 6

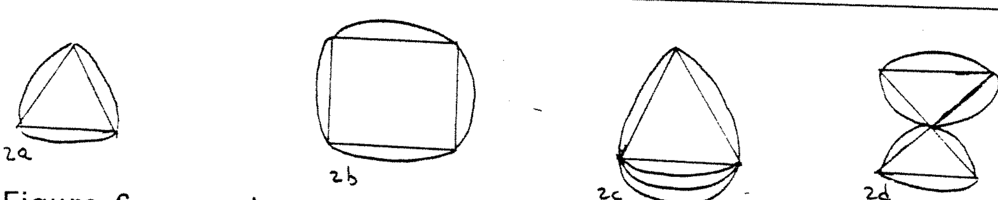

Figure 6: graphs entering the high temperature expansion after the ensemble average is performed

When points i and j are joined by 2n lines we attach a factor $\overline{\left(th \frac{\beta J_{ij}}{|i-j|^{\sigma}}\right)^{2n}}$ for the corresponding line, we then sum over all possible labelled graphs. As usual taking the logarithm eliminates all disconnected graphs from the sum. Since we are interested in the singular behaviour of the free energy we keep the most singular diagrams. These are the polygon graphs shown in fig. 7 .

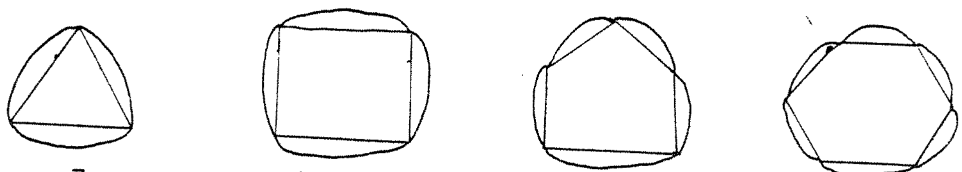

Figure 7: most singular graphs contributing to the free energy

We neglected diagrams having vertices connected by more than two lines like graph 2.c since the propagator associated with a multiply connected pair of points falls off much faster than the corresponding factor associated with a double bond. We also neglected, in this approximation, graphs having articulation points ,like graph 2d , since the sum over this class of graphs contains one lattice summation less than the sum over the class of graphs obtained by deleting the articulation point. These considerations are valid as long as the summations are slowly convergent, i.e. for $\sigma \simeq \frac{1}{2}$. The graphs in fig 7 are easily summed:

$$-\frac{1}{2}\left(\triangle + \square + \cdots\cdots\right)^2_{\text{connected}} = -\frac{1}{2}\sum_{n=3}^{\infty}\frac{1}{2n}\sum_{i_1\cdots i_n} g(i_1-i_2)\,g(i_2-i_3)\cdots g(i_n-i_1) \qquad 2.5$$

$$g(|k-l|) = \left(\operatorname{th}\frac{J_{kl}\beta}{|k-l|^{\sigma}}\right)^2 \equiv \int\frac{dx}{\sqrt{2\pi J^2}}\, e^{-\frac{x^2}{2J}}\left(\operatorname{th}\frac{\beta x}{|k-l|^{\sigma}}\right)^2 \qquad 2.6$$

The factor $\frac{1}{2n}$ enters because when $i_1 \cdots i_n$ run freely over the integers the graph $(i_1 i_2)(i_2 i_3)\cdots(i_n i_1)$ is counted 2n times. (Any cyclic permutation of $i_1 i_2 \cdots i_n$, or an inversion $i_1 \cdots i_n \to i_n \cdots i_1$ leaves the n-polygon graph invariant). To carry out the convolutions in eq. 2.5 we introduce the Fourier transform of g(j).

$$\hat{g}(k) = \sum_{n=-\infty}^{\infty} g(n)\, e^{ink} \qquad 2.7$$

$\hat{g}(k)$ will be evaluated for small k in the appendix to this section. Plancherel formula gives

$$\sum_{i_1 \cdots i_n} g(i_1 - i_2)\, g(i_2 - i_3) \cdots g(i_n - i_1) = \int_{-\pi}^{\pi} \frac{dk}{(2\pi)} \left[\hat{g}(k) \right]^n \qquad 2.8$$

We are interested in the singular part of the free energy so we add a finite number of terms (n=1 and n=2) to eq. 2.5 to carry out the sum explicitly. Combining this result with eq. 2.8 we find

$$-\beta F = N \left\{ \log 2 + \frac{(\beta J)^2}{2} \sum_{n=1}^{\infty} \frac{1}{|n|^{2\sigma}} \right\} + \frac{N}{4} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \log \left[1 - \hat{g}(k) \right] \qquad 2.9$$

The first term in eq. 2.9 is convergent provided $\sigma > \frac{1}{2}$ confirming the existence of the thermodynamical limit when $\sigma > \frac{1}{2}$. To analyse the second term we need the results of the appendix:

$$\hat{g}(k) = \beta^2 J^2 2\, \zeta(2\sigma) + 2\beta^2 J^2 x^{2\sigma-1} \sin\sigma\pi\, \Gamma(1-2\sigma) \qquad 2.10$$

From 2.9 and 2.10 we see the free energy is a continuous function of the temperature. Higher order derivatives diverge when $2\beta^2 J^2 \zeta(2\sigma) = 1$. The critical temperature $T_c = J\sqrt{2\zeta(2\sigma)}$ 2.11 decreases as σ increases. This can be understood qualitatively. As σ increases, the range of the interaction decreases, and fluctuations increase lowering the transition temperature.

The diagrams included in our calculation are the same as the ones considered by Thouless et.al.[1] But while their contribution for the infinite range model vanish in the thermodynamical limit, they give a finite contribution to the free energy per unit volume in our case. In the same approximation we can calculate the correlation function

$$\langle S_i S_j \rangle^2 \equiv Q(i-j) \qquad 2.12$$

This correlation function is related to the non linear susceptibility via linear response.

$$\chi_2 \equiv \frac{\partial^2 \chi}{\partial h^2} = \frac{1}{6} \overline{\left[\langle M^4 \rangle - 3 \langle M^2 \rangle^2 \right]}$$

$$= \frac{1}{6} \sum_{ijkl} \overline{\langle S_i S_j S_k S_l \rangle - 3 \langle S_i S_j \rangle \langle S_k S_l \rangle} \qquad 2.13$$

Assuming a symmetric distribution of bonds, or local gauge invariance 2.13 reduces to

$$\chi_2 = -\frac{1}{2} \sum_{ij} \overline{\langle S_i S_j \rangle^2}$$

Figure 8: Diagrams contributing to the non linear susceptibility. The double line represents a factor

Summing up the diagrams in fig. 8 we find:

$$Q(k,T) = \frac{\hat{g}(k)}{[1 - \hat{g}(k)]} \qquad 2.14$$

Using the results of the appendix we rewrite eq. 2.14 as

$$Q(k,T) = \left(\frac{T_c}{T}\right)^2 \frac{}{\left[1 - \left(\frac{T_c}{T}\right)^2 - 2\Gamma(1-2\sigma)(\sin\sigma\pi)\left(\frac{J}{T_c}\right)^2 k^{2\sigma-1}\right]} \qquad 2.15$$

When $T > T_c$, $Q(|i-j|, T)$ decays exponentially while at $T = T_c$.

$$Q(k, T_c) \simeq \frac{1}{k^{2\sigma-1}} \qquad 2.16$$

Notice that for $\sigma = 1$, $\int dk \; Q(k, T_c) = \infty$

suggesting that fluctuations destroy any type of ordering. Since our calculation is valid only for σ close to $\frac{1}{2}$ this conclusion should be taken with caution.

The non linear susceptibility diverges as :

$$Q(k=0, T) \simeq \frac{1}{T - T_c} \qquad 2.17$$

Therefore the critical index of the order parmeter susceptibilty is given by $\gamma = 1$.

The analysis above strongly suggests the presence of an instability when T falls below T_c, but it is too crude to predict the precise form of the singularities in the critical region or to determine the range of validity of the equations derived. In the next section we will perform a renormalization group analysis that will confirm the validity of eqs. 2.16 and 2.17 provided $\frac{1}{2} \leq \sigma \leq \frac{2}{3}$. It will also yield corrections to eq. 2.17 to first order in $\varepsilon = \sigma - \frac{2}{3}$.

APPENDIX

The function g(j) is defined by eq. 2.6

$$g(j) = \frac{\langle th^2 J_{ij}\beta\rangle}{|j|^\sigma} = \int \frac{dJ_{ij}}{\sqrt{2\pi J^2}}\, e^{-\frac{J_{ij}^2}{2J^2}} \left(th\, \frac{J_{ij}\beta}{|j|^\sigma}\right)^2$$

The behaviour of $\hat{g}(k)$ at small k is controlled by the long distance behaviour of g(j) so we will approximate it by

$$g(j) = \int \frac{dJ_{ij}}{\sqrt{2\pi}}\, e^{-\frac{J_{ij}^2}{2J^2}} \frac{(J\beta)^2}{|j|^{2\sigma}} \simeq \frac{\beta^2 J^2}{|j|^{2\sigma}}$$

$$\hat{g}(k) = \sum_{n=-\infty}^{\infty} g(n)\, e^{ink} \simeq 2\beta^2 J^2 \sum_{n=1}^{\infty} \frac{1}{|n|^{2\sigma}}\, e^{ink}$$

We substract the k independent term and estimate the first k dependent correction

$$\hat{g}(k) - \beta^2 J^2 2 \sum_{n=1}^{\infty} \frac{1}{|n|^{2\sigma}} = 2\beta^2 J^2 \sum_{n=1}^{\infty} \frac{\cos nk - 1}{|n|^{2\sigma}}$$

$$\hat{g}(k) \qquad \sum_{n=1}^{\infty} \frac{\cos nk - 1}{|n|^{2\sigma}} \simeq -2 \sum_{n=1}^{\infty} \frac{\sin^2 \frac{nk}{2}}{|n|^{2\sigma}} \simeq -\frac{k^{2\sigma-1}}{2^{2\sigma-2}} \int_0^\infty \frac{dx}{x^{2\sigma}} \sin^2 x$$

Using the definition of the Riemann zeta function, and evaluating the integral integral[2] we find

$$\hat{g}(k) = 2\beta^2 J^2 \zeta(2\sigma) + \beta^2 J^2 k^{2\sigma-1}\, \Gamma(1-\sigma) \sin\sigma\pi$$

Note that since $\frac{1}{2} < \sigma < 1$, $\Gamma(1-\sigma) \sin\sigma\pi < 0$

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3.3 MAPPING ONTO A CONTINUUM FIELD THEORY

In the study of critical phenomena it is extremely convenient to replace the original hamiltonian by an effective Ginzburg-Landau hamiltonian which describes the same physics on scales much larger than the microscopic scale.

The effective hamiltonian should be a functional of a coarse grained average of the order parameter. For the spin glass problem the Landau Ginzburg theory was first constructed by Harris Lubensky and Chen[1] Their results apply to the Edwards-Anderson (i.e. short range) spin glass model but their construction can be trivially generalized to include the effect of long range forces. In this section we will present this generalization which is essential to the determination of the upper critical dimensionality of the model, to be carried out in the next section.

Since the one dimensionality of the model is not needed in the following derivations we will consider the d dimensional generalization of eq 1.3 . Our hamiltonian is given by

$$H = \frac{1}{2} \sum_{i \neq j} \frac{J_{ij} S_i S_j}{|i-j|^{\sigma}}$$

S are Ising spins, J are independently distributed gaussian random variables with probability distribution

$$P(J_{ij}) = \frac{e^{-\frac{J_{ij}^2}{2J^2}}}{\sqrt{2\pi J^2}}$$

To perform the quenched average of the free energy we use the well known replica trick.[2] We average the nth moment of the partition function by considering n identical replicas of the original system, and in the end, we retrieve the physical quantities by letting the number of replicas go to zero. The configuration average of the nth moment of the partition function is given by

$$\overline{Z^n} = \int \prod_{i<j} P(J_{ij})\, dJ_{ij} \; \mathrm{tr} \exp \sum_{i<j} \sum_{\alpha=1}^{n} \frac{J_{ij} S_i^\alpha S_j^\alpha}{|i-j|^\sigma} \qquad 3.3$$

α is the replica index running from 1 to n. Carrying out the gaussian integral we find

$$\overline{Z^n} = \mathrm{tr}\, e^{\mathcal{H}_{eff}[S_i^\alpha]} \qquad 3.4$$

Where the effective hamiltonian is given by

$$\mathcal{H}_{eff}[S_i^\alpha] = \frac{\beta^2 J^2}{2} \sum_{i<j} \sum_{\alpha\beta} \frac{S_i^\alpha S_i^\beta S_j^\alpha S_j^\beta}{|i-j|^{2\sigma}} \qquad 3.5$$

The trace in eq. 3.4 is over the replicated spin variables S_α^i $\alpha = 1 \cdots n$. The order parameter in the replica formalism is identified with $\langle S^\alpha S^\beta \rangle$ where the average here is taken with respect to 3.5. Naively, one would like to **obtain** an effective hamiltonian in the variables

$$q_i^{\alpha\beta} \equiv S_i^\alpha S_i^\beta \qquad 3.6$$

Unfortunately, the $q^{\alpha\beta}$ as defined by eq. 3.6 are not independent variables. (There are $\frac{n(n-1)}{2}$ $q_{\alpha\beta}$ and only n S_α).

As emphasized by Anderson,[3] the variables $q^{\alpha\beta}$ defined by eq. 3.6 obey a positivity constraint

$$\mathrm{tr}\; q_i^p = \sum_{\alpha_1 \cdots \alpha_p} s^{\alpha_1} s^{\alpha_2} s^{\alpha_2} s^{\alpha_3} \cdots s^{\alpha_p} s^{\alpha_1} \geq 0 \qquad 3.7$$

This positivity is expected from the physical interpretation of the Edwards-Anderson order parameter as a spin autocorrelation function.

$$q = \lim_{t \to \infty} \langle s(0) s(t) \rangle \qquad 3.8$$

The spin variables and therefore the $q^{\alpha\beta}$ derived from them are discrete. In order to get rid of all these constraints we use the mathematical identity[4]

$$e^{-\frac{1}{2} b A^{-1} b} = \int \frac{dx}{(\det A)} e^{-\frac{1}{2} X A X} e^{x.b}$$

to obtain

$$\langle Z^n \rangle = \int \prod_{i\, \alpha<\beta} \mathcal{D} Q_i^{\alpha\beta} \exp -\frac{1}{2} \sum_{\substack{ij \\ \alpha<\beta}} K_{ij} Q_i^{\alpha\beta} Q_j^{\alpha\beta} \quad \mathrm{tr} \exp \sum_{i\, \alpha<\beta} Q_i^{\alpha\beta} s_i^\alpha s_i^\beta$$

$$= \int \mathcal{D} Q^{\alpha\beta} e^{-\mathcal{L}[Q]} \qquad 3.9$$

3.10

The matrix K_{ij} is defined by

$$\sum_{l \neq i} K_{il} \frac{}{|l - j|^\sigma} = \delta_{ij} \qquad 3.11$$

The lagrangian defined by eq. 3.10

$$\mathcal{L}[Q] = \frac{1}{2} \sum_{\substack{ij \\ \alpha\beta}} K_{ij} Q_i^{\alpha\beta} Q_j^{\alpha\beta} - \sum_i \log \mathrm{tr}_{s_\alpha} Q_i^{\alpha\beta} s^\alpha s^\beta \qquad 3.12$$

is equivalent to the initial effective hamiltonian 3.5, in the sense that

$$\mathrm{tr}\; e^{\mathcal{H}_{eff}[s^\alpha]} = \int \mathcal{D} Q_i^{\alpha\beta} e^{-\mathcal{L}[Q]} \qquad 3.13$$

and

$$\mathrm{tr}\; s_i^\alpha s_i^\beta e^{\mathcal{H}_{eff}[s]} = \int \mathcal{D}Q\, Q_i^{\alpha\beta} e^{-\mathcal{L}[Q]} \qquad 3.14$$

Furthermore, all the correlation functions of the replicated spins can be expressed in terms of correlation functions of the Q variables calculated with lagrangian 3.12 .

We exploit the translation invariance of the replicated system by fourier transforming eqs. 3.11 and 3.12 .

$$\sum_{ij} K_{ij}\, Q_i^{\alpha\beta}\, Q_j^{\alpha\beta} = \sum_k Q^{\alpha\beta}(k)\, Q^{\alpha\beta}(-k)\, K(k) \qquad 3.15$$

$$K(k) \equiv \left[\sum_j \frac{e^{ik\cdot ja}}{|j|^\sigma}\, \beta^2 J^2 \right]^{-1} \qquad 3.16$$

For small k

$$\sum_{j\neq 0} \frac{e^{ik\cdot ja}}{|j|^{2\sigma}} = C_1 - C_2\, |ak|^{2\sigma-d} \qquad 3.17$$

C_1 and C_2 are positive constants defined by

$$C_1 = \sum_{j\neq 0} \frac{1}{|j|^\sigma} \qquad C_2 = \int \frac{d^dx\,[1-e^{i\sum_{k=1}^{d} x_k j_k}]}{|x|^{2\sigma}} \qquad j \equiv (j_1, \dots j_d) \qquad 3.18$$

With these definitions we have, for small k

$$K(k) = \left[1 + \frac{C_2}{C_1}\, |ak|^{2\sigma-d} \right] \frac{1}{\beta^2 J^2 C_1} \qquad 3.19$$

The effective lagrangian can be split in two parts $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$

$$\mathcal{L}_0[Q] = \frac{1}{4} \sum_{ij} K_{ij}\, \mathrm{tr}\, Q_i Q_j = \frac{T^2}{4 J^2 C_1} \sum_k \left[1 + \frac{C_2}{C_1}\, |ak|^{2\sigma-d}\right] \mathrm{tr}\, Q(k)^2$$

$$= \frac{T^2}{4 C_1 J^2} \int \frac{d^dx}{a^d} \left\{ \mathrm{tr}\, Q(x)^2 + \frac{C_2}{C_1}\, a^{2\sigma-d}\, \mathrm{tr}\, \nabla^{\frac{2\sigma-d}{2}} Q\ \nabla^{\frac{2\sigma-d}{2}} Q \right\} \qquad 3.20$$

What is meant by a fractional power of the gradient becomes clear when written in momentum space as in eq 3.20 . To evaluate $\mathcal{L}_1$ we expand in powers of the field Q. We will keep up to cubic terms since higher powers of the field will turn out to be irrelevant.

$$e^{-\mathcal{L}_1[Q]} = \mathrm{tr}\,\exp \sum Q^{\alpha\beta} S^\alpha S^\beta = 1 + \sum_i Q_i^{\alpha\alpha} + \sum_{\alpha<\beta<\gamma} Q_i^{\alpha\beta} Q_i^{\beta\gamma} Q_i^{\gamma\alpha}$$

$$= \mathrm{const}\ \exp \int \frac{d^d x}{a^d} \left[\frac{1}{4}\,\mathrm{tr}\, Q^2(x) + \frac{1}{3!}\,\mathrm{tr}\, Q^3(x) \right] \qquad 3.21$$

Rescaling the field variables

$$\sqrt{\frac{C_2}{C_1^2}}\; a^{\sigma-d}\; \frac{T}{J} Q \rightarrow Q \qquad 3.22$$

and combining 3.21 and 3.22 we end up with the continuum field theory

$$\mathcal{L}[Q] = \mathrm{tr} \int d^d x \left\{ \frac{1}{4} \left(\nabla^{\frac{2\sigma-d}{2}} Q\right)^2 + \frac{\pi}{4} Q^2 - w Q^3 \right\} \qquad 3.23$$

$$\pi = a^{d-2\sigma} \frac{C_1}{C_2} \left[1 - C_1 \frac{J^2}{T^2} \right] \qquad 3.24$$

$$w = \frac{1}{3!} \left(\frac{J}{T}\right)^3 \frac{C_1^3}{C_2^{3/2}} \qquad 3.25$$

The effective hamiltonian 3.12 and its truncated expansion in powers of the field 3.23 are the main results of this section. Its physical meaning as a physical Landau free energy functional is not completely clear. The order parameter q from its very definition is a positive quantity and its fluctuations should be positive too. (For an excellent discussion of this point see ref. 3). On the other hand for the mathematical manipulations we have made to be correct the path integral over the Q fields (see for instance eq. 3.9) is unconstrained. Nevertheless the form of the free energy functional is such that it gives sensible (i.e. positive) values for all the averaged correlation functions. Having derived eq. 3.23, let us pause to examine each of its terms.

The cubic vertex is characteristic of the spin glass problem. It introduces new divergences in perturbation theory and consequently new terms in the renormalization group recursion relations to be discussed in section 3.4. It is essential for the positivity of all the expectation values of products of Q fields. To illustrate this point let us reproduce a heuristic argument originally due to Lubensky. Consider the Sherrington Kirkpatrick ansatz $q_{\alpha\beta} = q \;\; \alpha \neq \beta$. Inserting it in eq. 3.23 we find the free energy per degree of freedom

$$\mathcal{L}\,[Q^{\alpha\beta}] = \frac{r}{4}\, q^2 - \omega(n-2)\, q^3$$

When r>0 the only non negative minima is at $q=0$. See fig 9a. When r<0 a non trivial positive minima develops, as shown in fig. 9b.

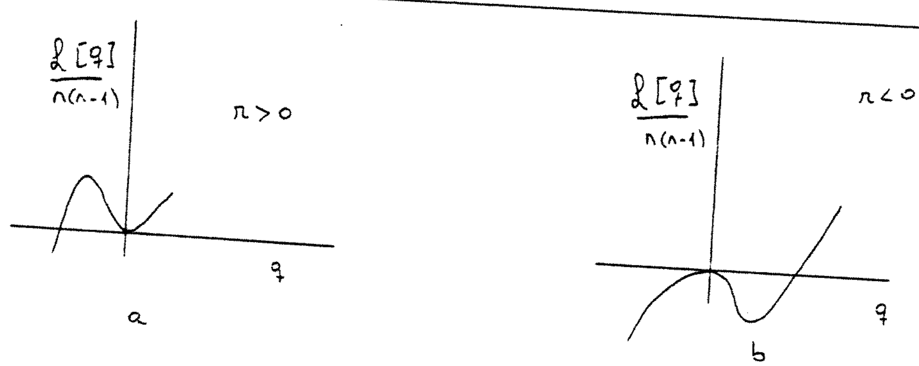

Figure 9:

Therefore for $n < 2$ the cubic term plays a very different role than in the usual Landau theory. For $T < T_c$, it continuously developes a positive minimum causing a continuous second order phase transition instead of a first order transition as one would normally expect.

The mass term r, in eq. 3.23, controls the distance from the critical point. When $T=T_c=\sqrt{C_1}\,J$ r vanishes, as $T < T_c$ r becomes negative indicating the instability of the paramagnetic phase. In one dimension

$$C_1 = \sum_{i \neq 0} \frac{1}{|i|^{2\sigma}} = 2\zeta(2\sigma) \qquad 3.27$$

and eq. 3.24 becomes

$$T_c = \sqrt{2\zeta(2\sigma)}\, J \qquad 3.28$$

in agreement with the result of section .

The $(\nabla^{\frac{2\sigma-d}{2}} Q)^2$ term replace the $(\nabla Q)^2$ characteristic of short range interactions. Fourier transforming it gives $k^{2\sigma-d} Q^2$ which, for small k, is much larger than the corresponding short range $k^2 Q$. Long range interactions supress fluctuations more efficiently than their short range counterparts extending the range of validity of mean field theory.

In the continuum theory the cubic coupling w has dimensions of $(\text{length})^{2d-3\sigma}$. Hence from naive dimensional analysis one would expect the upper critical range to be $\sigma = \frac{2d}{3}$. This will be confirmed by the renormalization group analysis of the following section.

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3.4 EXPANSION ABOUT THE UPPER CRITICAL RANGE

In this section we use the Landau Ginzburg hamiltonian derived in the previous section to analyse the critical behaviour of our system near the upper critical range. We determine the upper critical range to be $\sigma = \frac{2d}{3}$, we derive renormalization group recursion relations and calculate critical exponentes in an ε- expansion in the range $\sigma = \frac{2d+\varepsilon}{3}$.

The absence of a thermodynamical limit when $\sigma = \frac{d}{2}$, suggests mean field theory should be valid in its vicinity. To establish the range of values of the parameter σ for which fluctuations do not influence the critical behaviour we invoke the Ginzburg criterion. From the free energy functional we see that the energy cost of a fluctuation of the order parameter over the correlation length is

$$\int_{|x|\leq \xi} \left(\nabla^{\frac{2\sigma-d}{2}} Q\right)^2 d^d x = \frac{\xi^d Q^2}{\xi^{2\sigma-d}} \qquad 4.1$$

From eq. 3.22, $Q \sim r$ while from dimensional considerations $\xi^{-2\sigma+d} \sim r$. Inserting this in eq. 4.1 we find

$$\int_{|x|\leq \xi} \left(\nabla^{\frac{2\sigma-d}{2}} Q\right)^2 d^d x = \xi^{4d-6\sigma} \qquad 4.2$$

That is, when $\sigma \leq \frac{2d}{3}$ fluctuations over the relevant length cost infinite energy and hence are heavily suppressed, while for $\sigma \geq \frac{2d}{3}$ these fluctuations cost little energy and dominate the critical behaviour as $T \to T_c$ and $\xi \to \infty$. These considerations indicate that the upper critical range for this problem is $\sigma = \frac{2d}{3}$.

Now we turn to the derivation of the renormalization group recursion relations for the Landau-Ginzburg effective hamiltonian.

This analysis for $\sigma = \frac{d}{2}+1$, i.e. for short range forces, was carried out by Harris et al.[1] We will carry out their analysis in the presence of a long range force term. Because of this term the spin rescaling factor is determined by the $(\nabla^{\frac{2\sigma-d}{2}}Q)^2$ term to all orders in the coupling w. This is a general feature of long range forces.[2]

The quadratic part of the hamiltonian defines the free propagator

$$\langle Q_{\alpha\beta}(k)\, Q_{\alpha'\beta'}(k') \rangle = G_0(k)\, \delta_{k+k'}\, \delta_{\alpha\beta;\,\alpha'\beta'} \qquad 4.3$$

$$G_0(k) = \frac{}{[k^{2\sigma-d} + r]} \qquad 4.4$$

The recursion relations for the coupling are found by integrating out wavevectors in the range $\frac{1}{s} \leq k \leq 1$ (we have chosen units so as to have a cuttoff of order unity), rescaling the field Q, $Q \to \lambda_s Q$ and changing the length scale so that the rescaled wavectors lie in the range $0 \leq k \leq 1$. The rescaling factor λ_s is determined so as to make the coeeficient of $(\nabla^{\frac{2\sigma-d}{2}}Q)^2$ invariant under the renormalization group transformation.

Under rescaling

$$k^{2\sigma-d} \to \frac{k^{2\sigma-d}}{s^{2\sigma-d}} \qquad 4.5$$

hence we chose

$$\lambda_s = s^{\frac{2\sigma-d}{2}} \qquad 4.6$$

All the integrals entering the recursion relations are well behaved since $\frac{1}{s}$ provides an infrared cuttoff. Hence the renormalization group does not generate non analytic terms like $k^{2\sigma-d}$. Eq. 4.6 is correct to all orders in the coupling w, provided σ is small enough to be in the domain of attraction of the long range fixed point.[3]

The cubic vertex w renormalizes r and w. The relevant Feynman graphs are shown in fig. 10

Figure 10: Graphs renormalizing the temperature and the cubic vertex

The lines represent the free propagator and the vertices are generated by the cubic term in 3.17. Their contribution to the renormalized hamiltonian is given by

$$\delta \mathcal{H}_1 = \frac{1}{4} \sum_{k<\frac{1}{s}} \left[w^2 (n-2)\, 36 \int_{\frac{1}{s}<k'<1} \frac{d^d k'}{(2\pi)^d}\, G_0(k+k')\, G_0(k') \right] \mathrm{tr}\, Q(k)^2 \qquad 4.7$$

$$\delta \mathcal{H}_2 = \frac{w^3}{\sqrt{V}} \sum_{k_1 k_2 k_3<\frac{1}{s}} \mathrm{tr}\, Q(k_1) Q(k_2) Q(k_3) \left[3^3\, 2^3 (n-2) \int_{\frac{1}{s}<k'<1} \frac{d^d k'}{(2\pi)^d}\, G_0(k')\, G_0(k'+k_2)\, G_0(k'-k_1) \right] \qquad 4.8$$

The spin rescaling factor yields immediatly the exponent[4]

$$\eta = 2 + d - 2\sigma \qquad 4.9$$

Using eqs. 4.7 – 4.8 and the rescaling factor we finally obtain the recursion relations

$$r' = s^{2-\eta} \left\{ r - 36(n-2)\, w^2 \int_{1/s}^{1} \frac{S_d\, q'^{d-1}\, dq'}{[q'^{2\sigma-d} + r]^2} \right\} \qquad 4.9$$

$$w' = s^{(6-3\eta-d)/2} \left\{ w + 36(n-2)\, w^3 \int_{1/s}^{1} \frac{dq'\, S_d\, q'^{d-1}}{[q'^3]^{2\sigma-d}} \right\} \qquad 4.10$$

Setting $s = e^{l}$ we convert the discrete recursion relations into differential equations

$$\frac{dr}{dl} = (2-\eta)\, r - \frac{36(n-2)\, w^2 S_d}{[1+r]^2} + O(w^3) \qquad 4.11$$

$$\frac{dw}{dl} = \left(\frac{6-3\eta-d}{2}\right) w + 36(n-2)\, w^3 S_d + O(w^4) \qquad 4.12$$

Inserting the known value of η from eq. 4.9 and writing $\sigma = \frac{2d+\varepsilon}{3}$ in eq. 4.12 we find

$$\frac{dw}{dl} = \varepsilon w + 36(n-2)\, w^3 S_d + O(w^4) \qquad 4.13$$

confirming the cubic term is irrelevant when $\sigma < \frac{2d}{3}$ (i.e when $\varepsilon < 0$). Hence $\sigma = \frac{2d}{3}$ is the upper critical range of our problem in agreement with the qualitative arguments of the previous section.

For $\varepsilon < 0$ the gaussian fixed point $r=0$, $w=0$ is stable and the exponent ν is given by its classical value:

$$\frac{1}{\nu} = 2 - \eta = 2\sigma - d \qquad 4.14$$

Using the scaling relation $\gamma = \nu(2-\eta)$, we can predict the divergence of the non linear susceptibility

$$Q(k=0, T) \propto \frac{1}{[T - T_c]}$$

This agrees with the results of the previous section (see eq. 2.17), but now we know those results are valid provided $\sigma < 2/3$. For $\varepsilon > 0$, w is strongly relevant. For small ε we can find a fixed point with w of order ε and calculate critical exponents in an ε expansion. The new fixed point is

$$w^{*2} = -\frac{\varepsilon}{36(n-2)K_d} \qquad 4.15$$

$$r^{*} = -\frac{3}{d}\varepsilon \qquad 4.16$$

Linearizing eqs. 4.11 - 4.12 around the new fixed point we find

$$\delta\dot{r} = \left(\frac{d}{3} + \frac{2\varepsilon}{3}\right)\delta r - \frac{2\varepsilon}{(1+r^{*})}\delta r + \frac{\delta w}{[1+r^{*}]^2}\frac{2\varepsilon}{w^{*}} \qquad 4.17$$

$$\delta\dot{w} = \varepsilon\,\delta w - 3\varepsilon\,\delta w \qquad 4.18$$

The eigenvalues of the linearized equations are -2ε and $\frac{d-4\varepsilon}{3}$. This gives the desired corrections to ν.

$$\frac{1}{\nu} = \frac{d-4\varepsilon}{3} \qquad 4.19$$

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3.5 EXPANSION ABOUT THE LOWER CRITICAL RANGE

In this section we discuss the critical behaviour of the model near its lower critical range. We determine the value of the lower critical range to be $\sigma=1$, from a simple physical argument first, and then from a more rigourous analysis using a renormalization group. To investigate the critical behaviour we map the random model onto a classical, translation invariant, spin chain with long range interactions. We then apply the formalism developped in section 2.4 to this problem.

Most of the theoretical work on spin glasses has concentrated on the Sherrington-Kirkpatrick model which is a model effectively embedded in an infinite dimensional lattice. The value of the lower critical dimensionality of a short range Edwards-Anderson model is still unknwon although most of the theoretical work seems to converge on d=4 while the experimentalists insist they observe a sharp phase transition in three dimensions. (See ref. 8 in section 3.1). The relevance of the mean field results to three dimensional materials is again another open question. In this section we analyse our spin glass model near its lower critical range and this is, to the best of our knowledge, the first attempt to study the critical behaviour of a spin glass near its lower critical dimensionality.

The critical exponents we find in this section differ radically from the ones found in the previous one. In particular the specific heat exponent $\alpha=-\infty$ indicating a perfectly smooth specific heat. These results suggest that the Sherrington Kirkpatrick model might not be a good guide for understanding the finite dimensional world. The results here obtained depend crucially on two assumptions: 1)That the ground state of the system is $q_{EA}=1$ and fluctuations in the value of this order parameter are the relevant excitations that disorder the system. 2) That the symmetric fixed point we found is accessible from the initial conditions and hence effectively controls the critical behaviour. We end this section with a discussion of these assumptions.

We start with model 1.3. The hamiltonian is given by

$$H = \sum_{ij} \frac{\sigma_i \sigma_j J_{ij}}{|i-j|^{\sigma}} \tag{5.1}$$

Here σ_i are Ising spins and J_{ij} are independent random variables with probability distribution

$$P(J_{ij}) = \frac{e^{-\frac{J_{ij}^2}{2J^2}}}{\sqrt{2\pi J^2}} \tag{5.2}$$

The lower critical range can be determined from the following arguments.[1]

Consider one (out of the many possibly degenerate) ground states of the random system. Denote it by $S_i^0(J_{kl})$, indicating explicitly its dependence on the quenched disorder. $S_i^0(J)$ is for example a solution of the mean field equations indicating its stability against single spin flips.

$$S_i^0(\{J_{kl}\}) = \operatorname{sign} \sum_k J_{ik} S_k^0(J) \tag{5.3}$$

Clearly J_{ij} and $S_i^0(J)$ are strongly correlated. Lets consider fluctuations about this ground state and perform a Mattis Luttinger[2] transformation:

$$S_i = S_i^0(J)\, s_i \tag{5.4}$$

The hamiltonian in the new variables s_i is given by

$$\sum_{ij} \frac{S_i^o(J)\, J_{ij}\, S_j^o(J)}{|i-j|^\sigma}\, s_i s_j \qquad 5.5$$

In the appendix to this section we show that the random variables

$$K_{ij} = \frac{S_i^o(J)\, J_{ij}\, S_j^o(J)}{|i-j|^\sigma} \qquad 5.6$$

have a positive expectation value

$$\overline{K_{ij}} \simeq \frac{\text{const}}{|i-j|^{2\sigma}} \qquad 5.7$$

Hence as far as fluctuations about a single ground state our model is equivalent to a ferromagnetic chain with $\frac{1}{|i-j|^{2\sigma}}$ interacions. The interaction between the defects (now defects are defined with respecto to the random configuration) is proportional to $|i-j|^{2-2\sigma}$. The lower critical range for this problem is known to be[3] $\sigma = 1$.

To give a more rigourous argument for the value of the lower critical range and to give a quantitative estimate of the critical exponents we turn to a renormalization group that exploits the one dimensionality of the model. The nth moment of the partition function is easily calculated

$$\overline{Z^n} = \int P(J_{ij})\, dJ_{ij}\, Z^n = \operatorname{tr} \exp - H_n[S_\alpha] \qquad 5.8$$

$$-H_n = (\beta J)^2 \sum_{i \neq j} \left(\sum_{\alpha=1}^{n} \sigma_i^\alpha \sigma_j^\alpha \right)^2 \qquad 5.9$$

As in section 3.3 we introduced n replicas of the system. The effective hamiltonian in eq. 5.2 can be rewritten as

$$-H_n = \sum_{i<j} \frac{2K(\sigma_i \sigma_j)}{|i-j|^\sigma} \qquad 5.10$$

$$K(\sigma_i \sigma_j) = (\beta J)^2 \left[\left(\sum_{\alpha=1}^{n} \sigma_i^\alpha \sigma_j^\alpha \right)^2 - n^2 \right] \qquad 5.11$$

In eqs. 5.10-5.11 we added a constant to the free energy so that K obeys the condition

$$K(\sigma_i\ \sigma_j) = 0 \quad \text{if} \quad \sigma_i = \sigma_j \qquad 5.12$$

We interpret $\{\sigma_i^\alpha\}$ as a spin variable that can be in 2^n different states. Each state is an n dimensional vector with entries equal to 1 or -1 . This states can be visualized as vertices of an n-dimensional hypercube with center at the origin. To clarify the notation we give an example for n=3.

For n=3 there are $2^3 = 8$ states we denote by $S_\alpha \quad \alpha = 1 \cdots 8$

$$\begin{array}{ll} S_1 = (1\ 1\ 1) & S_5 = (1\ 1\ -1) \\ S_2 = (1\ -1\ 1) & S_6 = (1\ -1\ -1) \\ S_3 = (-1\ 1\ 1) & S_7 = (-1\ 1\ -1) \\ S_4 = (-1\ -1\ 1) & S_8 = (-1\ -1\ -1) \end{array} \qquad 5.13$$

The interaction between spin i in state S_{α_i} and spin j in state S_{α_j} is given by

$$\frac{K(S_{\alpha_i}\ S_{\alpha_j})}{|i-j|^\sigma} = \frac{\beta^2 J^2 (S_{\alpha_i} \cdot S_{\alpha_j})^2}{|i-j|^\sigma} \qquad 5.14$$

$S_{\alpha_i} \cdot S_{\alpha_j}$ is the standard dot product of three dimensional vectors. Sometimes we will abbreviate

$$K(S_\alpha\ S_\beta) = K(\alpha, \beta) \qquad 5.15$$

The original couplings have the symmetry

$$K(\pi(S_\alpha), \pi(S_\beta)) = K(S_\alpha\ S_\beta) \qquad 5.16$$

Here, $\pi(S_\alpha)$ is the vector obtained by applying the permutation π to its components. Geometrically the interaction between two spins depends only on the relative position between the spins on the hypercube, or in other words $K(S_\alpha, S_\beta)$ is invariant under the point group of the hypercube. This symmetry is preserved under renormalization. The renormalization group equations 4.30–4.32 of chapter II, in zero field, become

$$\frac{d}{dl} y(\alpha,\beta) = y(\alpha,\beta)\left[1 + \frac{2}{(\sigma-1)} K(\alpha,\beta)\right] + \sum_{\upsilon} y_{\alpha\upsilon} y_{\upsilon\beta} \qquad 5.17$$

$$\frac{d}{dl} K(\alpha,\beta) = (2-\sigma) K(\alpha,\beta) - \sum_{\upsilon} y^2_{\alpha\upsilon}\left[K(\alpha,\beta) + K(\alpha,\upsilon) - K(\beta,\upsilon)\right] \qquad 5.18$$
$$- \sum_{\upsilon} y^2_{\beta\upsilon}\left[K(\alpha,\beta) + K(\beta,\upsilon) - K(\alpha,\upsilon)\right]$$

The variables υ run over 2^n states on the hypercube. This equations have a fixed point

$$y^2_{\alpha\beta} = y^* = \frac{\varepsilon}{2^n+1} \qquad \alpha \neq \beta \qquad 5.19$$

$$K_{\alpha\beta} = K^* = \frac{1}{2}\left[-1 + (2-2^n)\sqrt{\frac{\varepsilon}{2^n+1}}\right] \quad \alpha\neq\beta \qquad 5.20$$

We now linearize eqs. 5.17 - 5.18 around the fixed point 5.19 - 5.20.

$$\delta \dot{y}_{\alpha\beta} = 2y^* \delta K_{\alpha\beta} + (2-2^n) y^* \delta y_{\alpha\beta} + 2y^* \sum_{\upsilon\neq\alpha,\beta} \delta y_{\alpha\upsilon} \qquad 5.21$$

$$\delta \dot{K}_{\alpha\beta} = 4y^* \delta y_{\alpha\beta} + 2y^* \sum_{\upsilon\neq\alpha,\beta} \delta y_{\alpha\upsilon} \qquad 5.22$$

This is a system of $2(2^n-1)$ linear equations. Since $y_{\alpha\beta}$ and $K_{\alpha\beta}$ depend only on the relative position of α and β. We fix α and let β vary over the 2^n-1 remaining points on the hypercube. This way we obtain 2^n-1 fugacities $y_{\alpha\beta}$ and 2^{n-1} couplings $K_{\alpha\beta}$, a total of $2(2^n-1)$ equations.

The linearized renormalization group transformation (in units of y^*) can be written in matrix form as

$$\begin{bmatrix} 0 & 4 & 0 & 2 & 0 & 2 & \cdots \\ 2 & 1 & 0 & 2 & 0 & 2 & \cdots \\ 0 & 2 & 0 & 4 & 0 & 2 & \cdots \\ 0 & 2 & 2 & 1 & 0 & 2 & \cdots \\ 0 & 2 & 0 & 4 & 0 & 4 & \cdots \\ 0 & 2 & 2 & 1 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \qquad 5.23$$

Due to the high symmetry of this matrix we can find the most general form of its eigenvectors

$$\delta y_{\alpha\beta} = (a, b, c, d, c, d, \cdots\cdots)$$
$$\delta K_{\alpha\beta} = (a, b, c, d, c, d, \cdots) \qquad 5.24$$

Other eigenvectors are obtained by interchanging any of the 2^n-2 pairs (c,d) with (a,b). Inserting the ansatz 5.24 into eqs. 5.25 we obtain

$$4b + (2^n-2)\,2d = \lambda a$$
$$2a + b + (2^n-2)\,2d = \lambda b$$

$$2a + 4d + (2^n - 3)\, 2d = \lambda c \qquad 5.25$$

$$2b + 2c + d + (2^n - 3)\, 2d = \lambda d$$

This set of equations has only two eigenvalues

$$\lambda_1 = 1.56 \qquad \lambda_2 = -2.56 \qquad 5.26$$

We do not know what is the physical meaning of the eigenvectors as we do the analytic continuation to n=0 and we will come back to this point when we discuss the accessibility of the fixed point. We assume the eigenvalues retain their usual meaning. The presence of a single positive eigenvalue indicates the existence of a single relevant variable in the absence of a magnetic field. Reinserting the factor y^* in front of equation 5.23 we find

$$\frac{1}{\nu} = 1.56\, y^* = 1.1\sqrt{\varepsilon} \qquad 5.27$$

As usual, ν is the exponent characterizing the divergence of the correlation length as $T \to T_c$.

$$\xi \simeq \frac{1}{(T - T_c)^{\nu}} \qquad 5.28$$

From the scaling relations

$$\alpha = 2 - d\nu = 2 - \frac{.9}{\sqrt{\varepsilon}} \qquad 5.29$$

as $\varepsilon \to 0$, $\alpha \to -\infty$ indicating a smooth specific heat.

All this analysis is valid for $\varepsilon > 0$. When $\varepsilon < 0$ eq. 5.18 indicates K always flows towards zero. Since $K \sim \frac{1}{T}$ the system always flows to infinite temperature and is disordered at any finite temperature. This analysis predicts the lower critical range of the one dimensional spin glass model with power law interaction is $\sigma = 1$.

Let us now discuss the major assumptions made in the derivation of our results. The renormalization group equations are based on a low fugacity expansion about an ordered state. For finite n, our chain picks a ground state (one of the vertices of the hypercube). The relevant fluctuations at low temperatures are isolated single spin flips. Since single spin flips cannot destroy the magnetization we make a partial summation of single flips to overturn finite domains. The renormalization group describes how overturning this domains renormalizes the magnetization to zero at the critical point. Then, the renormalization-group machinery yields the critical exponents.

The relevant question is whether it is valid to naively analytically continue, as we did, the results obtained for finite n to n=0. We do not have a complete answer to this question but we will discuss some bits of insight into this problem. While it is clear that a spin glass has a macroscopic number of degenerate ground states due to the frustration effect, it is perfectly possible that all of them correspond to a single ground state of the replicated spin system. This is, in replica space $q^{\alpha\beta}=1$ can be a good description of the ground state of a spin glass. In fact, all the mean field theories (Sommers, Parisi and Sompolinsky solutions of the Sherrington-Kirkpatrick model) predict that at T=0, $q^{\alpha\beta}(T)=1$. Picking a single state on the hypercube, say (111111), is equivalent to expanding about $q^{\alpha\beta}=1$.

The next question is whether the expansion in the number of flips, which is a perfectly sensible expansion in powers of $y \sim e^{-\frac{J}{T}}$ for finite n, can be analytically continued to n=0 to give some expansion in powers of $\frac{1}{T}$. The answer to this question is negative. We have performed the analytic continuation of this expansion up to second order. The zero and first order of the expansion for $q(T)$ give

$$q(T) = 1 - \frac{2J}{T\pi}$$

Unfortunately the second order term is of order one, showing that an analytic continuation of a expansion in the number of flips does not reduce to an expansion in powers of T as the number of replicas goes to zero. I believe the analytic continuation of the renormalization group equations is still valid even when the low temperature expansion on which it is based breaks down as n goes to zero. But to prove or disprove this assumption one needs an approach not involving the replica trick and this remains to be found.

The second major assumption made in the course of our analysis is that the symmetric fixed point 5.19-5.20 controls the critical behaviour of the system. This assumption looks rather safe since all the directions in parameter space correspond to the negative eigenvalue λ_2, with the exception of the direction associated with λ_1, the positive eigenvalue representing temperature, which of course is a relevant variable. In other words any initial condition near criticality flows close to the symmetric fixed point we found. This conclusion, reasonable as it seems, is based on a local analysis around the symmetric fixed point. Parameter space, in the limit $n \to 0$, is a zero dimensional space and we do not understand its topological properties. We cannot exclude more esoteric possibilities, like the existence of other attractive fixed points or runaway solutions, without performing a global analysis of the system 5.17-5.18. This is an extremely difficult problem.

APPENDIX

We want to calculate the average

$$K(i,j) = \left\langle \frac{S_i^0(J)\, S_j^0(J)\, J_{ij}}{|i-j|^{\sigma}} \right\rangle$$

Using replicas and introducing an external source coupled to the bonds this can be written as

$$K(i,j) = \lim_{n\to 0} \frac{\partial}{\partial l_{ij}} \overline{\mathrm{tr}\, e^{H_n[J_{ij}] + \sum \frac{l_{ij} J_{ij}}{|i-j|^{\sigma}}}}$$

$$= \lim_{n\to 0} \frac{\beta^2 J^2}{|i-j|^{2\sigma}} \sum_{\beta} \mathrm{tr}\, s_i^{\beta} s_j^{\alpha} s_i^{\alpha} s_j^{\alpha}\, e^{H_n}$$

$$= \frac{\beta^2 J^2}{|i-j|^{2\sigma}} \lim_{n\to 0} \left\{ 1 + \sum_{\beta\neq\alpha} \langle s_i^{\beta} s_i^{\alpha} \rangle \langle s_j^{\beta} s_j^{\alpha} \rangle \right\}$$

To evaluate the sum over the correlation functions we need to assume a specific mean field solution. This specific assumtion enters only in a multiplicative coeficient in front of the correlation function and does not enter the dependence on |i-j|. For definiteneess let us insert Parisi's solution[4]

$$\sum_{\beta\neq\alpha} (q^{\alpha\beta})^2 = -\int_0^1 q^2(x)\, dx$$

to obtain

$$K(i,j) = \frac{\beta^2 J^2}{|i-j|^{2\sigma}} \left\{ 1 - \int_0^1 q(y)\, dy \right\}$$

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3.6 SUMMARY

In this chapter we studied a one dimensional random chain with $\frac{1}{r^\sigma}$ interactions for different values of the parameter σ controlling the range of the interaction. It is instructive to contrast the results we obtained with the corresponding results for the (non random) ferromagnetic chain. See fig. 11.

The ferromagnetic models are well defined (i.e the thermodynamical limit exists) for $\sigma > 1$. This can be easily understood by considering the ground state energy per spin

$$E = \sum_{i=1}^{L} \frac{1}{|i|^\sigma}$$

which is finite as $L \to \infty$ for $\sigma > 1$.

The effect of the randomness is to lower the energy per spin so that even when the sum

$$\sum_{i=1}^{L} \frac{1}{|i|^{\frac{1}{2}+\varepsilon}}$$

is divergent, the random exchange J_{ij} produces enough cancellations so as to make the ground state energy per spin

$$E = \sum_{i=1}^{L} \frac{J_{oi}}{|i|^{\frac{1}{2}+\varepsilon}}$$

converge, as $L \to \infty$, for $\varepsilon > 0$.

The upper critical range of the ferromagnetic chain[1] is $\sigma = \frac{3}{2}$, and mean field theory is valid in the range $1 < \sigma < \frac{3}{2}$. The randomness of the interactions enhances fluctuations and the classical regime is attained for much smaller values of σ (i.e. for a much longer range of the interaction). For the random system mean field theory is valid for $\frac{1}{2} < \sigma < \frac{2}{3}$.

The lower critical range of the ferromagnetic chain is $\sigma = 2$. This can be understood from energy versus entropy considerations. In a chain of length L the energy of a domain wall is proportional to

$$\sum_{i=1}^{L} \sum_{j<i} \frac{1}{|j|^{\sigma}} \sim L^{2-\sigma}$$

The wall can be placed in L sites giving an entropy contribution

$$-T \log L$$

to the free energy. If $\sigma > 2$ (and for long enough chains) the entropy term dominates at all temperatures. It is always favourable to create domain walls, and the chain is disordered at all temperatures.

In random systems fluctuations are more important than in non random ones. The lower critical range is reduced from the ferromagnetic value to $\sigma = 1$. This can be understood from the results of the appendix to section 3.5. Defects defined as fluctuations from any ground state of the disordered system interact via $r^{2-2\sigma}$ power law forces. From a different perspective, we showed in section 3.5 that the defects in replica space that disordered the Edwards-Anderson order parameter had $r^{2-2\sigma}$ interactions. For $\sigma = 1$ this reduces to a gas of defects with logarithmic interactions, corresponding to the marginal dimension.

In the region $\sigma > 1$ the random system does not exhibit symmetry breaking while the corresponding pure system does. This fact has extremely interesting consequences like the presence of Griffiths[2] singularities in the magnetic field dependence of the free energy. These singularities come from bond configurations having large ferromagnetic regions. These configurations occur with very low probability but, when the pure system has spontaneous magnetization, give anomalously large contributions to the magnetization of the random system in the presence of a magnetic field. These singularities are characteristic of random systems and have no analog in the corresponding pure model.

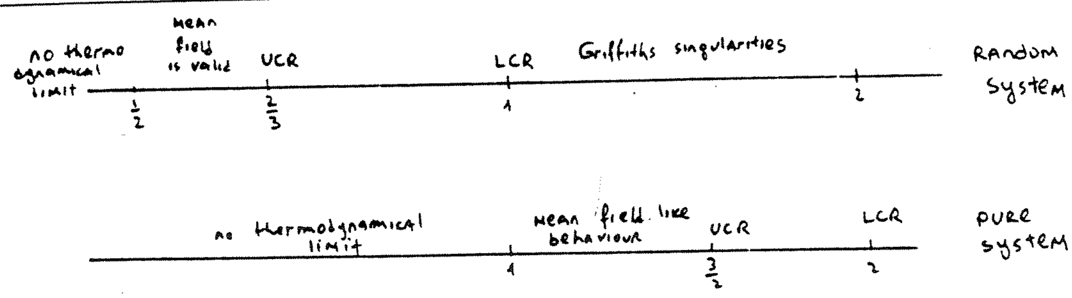

Figure 11: Random and pure system for different ranges of .

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