Solution to Problem Set 4

A spinless one dimensional electron gas of right movers is described by the Hamiltonian,

$$\mathcal{H}_0 = \sum_k k a_k^\dagger a_k = \int \psi^\dagger(x) \frac{\partial}{i\partial x} \psi(x)$$

The electron gas is interacting with a deep core electron (a single degree of freedom) described by,

$$\mathcal{H}_1 = E_f f^\dagger f$$

The interaction between the conduction electrons and the f level is given by,

$$\mathcal{H}_2 = v\psi^\dagger(0)\psi(0)\left(f^\dagger f - 1\right) = h_2\left(f^\dagger f - 1\right)$$

where $h_2 = \frac{v}{L}\sum_{k,k'} a_k^\dagger a_{k'}$.

1. First we will bosonize the fermionic degrees of freedom corresponding to the conduction electrons. For this we define the density operator, $\rho(q) = \sum_k a_{k+q}^\dagger a_k$, and notice that for large cutoff the density operators satisfy the commutation relation,

$$[\rho(q), \rho(-q')] = -\left(\frac{q'L}{2\pi}\right)\delta_{q,q'} \tag{1}$$

Thus, we can replace the density operators by bosonic operators defined by,

$$\begin{aligned} \rho(q) &= \sqrt{\frac{qL}{2\pi}} b_q^\dagger, \quad q > 0 \\ \rho(-q) &= \sqrt{\frac{qL}{2\pi}} b_q, \quad q > 0 \end{aligned} \tag{2}$$

so that the above commutation becomes the cannonical commutation relation of the bosonic degrees of freedom.

Now we examine the commutation relations between the density operators and the Hamiltonian $\mathcal{H}_0$. Again, for large values of the cutoff one can show

$$\begin{aligned} [\mathcal{H}_0, \rho(q)] &= q\rho(q) \\ [\mathcal{H}_0, \rho(-q)] &= -q\rho(-q) \end{aligned}$$

so that the Hamiltonian can be expressed as,

$$\mathcal{H}_0 = \frac{2\pi}{qL}\sum_{q>0} \rho(q)\rho(-q)$$

or, in terms of the bosonic variables, as,

$$\mathcal{H}_0 = \sum_{q>0} q b_q^\dagger b_q \tag{3}$$

The Fourier transform of the density operator is defined as $\rho(x) = \sum_q e^{iqx}\rho(q)$, and using eqn(2) it can be expressed in terms of the bosonic operators as,

$$\rho(x) = \frac{1}{L}\sum_{q>0}\sqrt{\frac{qL}{2\pi}}\left(b_q^\dagger e^{iqx} + b_q e^{-iqx}\right)$$

Then we have,

$$\psi^\dagger(0)\psi(0) = \rho(x=0) = \frac{1}{L}\sum_{q>0}\sqrt{\frac{qL}{2\pi}}\left(b_q^\dagger + b_q\right)$$

Thus, the full Hamiltonian can be written as,

$$\mathcal{H} = \sum_{q>0} q b_q^\dagger b_q + E_f f^\dagger f + v\left(f^\dagger f - 1\right)\frac{1}{L}\sum_{q>0}\sqrt{\frac{qL}{2\pi}}\left(b_q^\dagger + b_q\right) \tag{4}$$

2. For convenience we first solve part (c).

   We want to find a canonical transformation U that transforms $\mathcal{H}_0$ into $\mathcal{H}_0 - h_2$, i.e. $U^\dagger \mathcal{H}_0 U = \mathcal{H}_0 - h_2$. We can write the unitary operator U as e^{iA}, where A is a hermitian operator. Expanding $e^{iA}\mathcal{H}_0 e^{-iA}$ we find that, for each q mode we want $i\left[A, q b_q^\dagger b_q\right] = \frac{v}{L}\sqrt{\frac{qL}{2\pi}}\left(b_q^\dagger + b_q\right)$, and $\left[A, \left[A, q b_q^\dagger b_q\right]\right]$ to be a constant. This is identical to solving the problem of the displaced harmonic oscillator. The form of A that satisfies the above conditions is $i\sum_{q>0}\sqrt{\frac{2\pi}{qL}}\left(b_q^\dagger - b_q\right)$. Thus, we identify A to be of the form $\alpha\theta(0)$, where α is a constant and $\theta(x)$ is the canonical conjugate operator to $\rho(x)$, and is given by,

$$\theta(x) = i\sum_{q>0}\sqrt{\frac{2\pi}{qL}}\left(b_q^\dagger e^{iqx} - b_q e^{-iqx}\right)$$

   It is easy to find α, and we finally have,

$$U = e^{i\frac{v}{2\pi}\theta(0)} \tag{5}$$

   The constant $\left[A, \left[A, q b_q^\dagger b_q\right]\right]$ is interesting to calculate since it will eventually give shift in the f level energy (i.e. self energy correction). The final result for the unitary transformation is,

$$U^\dagger \mathcal{H}_0 U = \mathcal{H}_0 - h_2 - \delta \tag{6}$$

   where $\delta = \frac{v^2}{4\pi}$.

3. Now we look at part (b).

   Since $\left[f^\dagger f, \mathcal{H}\right] = 0$, we can describe the eigenstates of the Hamiltonian when $f^\dagger f = 0$, and when $f^\dagger f = 1$. In the $f^\dagger f = 1$ sector, the Hamiltonian is $\mathcal{H}_0 + E_f$. The eigenstates are those of a free electron gas, and the eigenvalues are those corresponding to the free electron eigenstates plus the constant energy E_f.

In the $f^\dagger f = 0$ sector, the Hamiltonian is $\mathcal{H}_0 - h_2$. Since $\mathcal{H}_0$ and $\mathcal{H}_0 - h_2$ are connected by a canonical transformation, the eigenstates of the latter are of the form $U^\dagger \,|\, \phi \,\rangle$ for every eigenfunction $|\, \phi \,\rangle$ of $\mathcal{H}_0$. The corresponding eigenvalue remains the same except without the term E_f.

Since $E_f << 0$, the ground state of the system is the filled Fermi sea plus the f level occupied. We can denote it as $|\, 0 \,\rangle = |\, FS \,\rangle \otimes |\, 1 \,\rangle$.

4. The Green's function of the heavy hole is given by,

$$\begin{aligned} G(t) &= \langle\, 1 \,| \otimes \langle\, FS \,|\, e^{i\mathcal{H}t} f^\dagger e^{-i\mathcal{H}t} f \,|\, FS \,\rangle \otimes |\, 1 \,\rangle \\ &= e^{iE_f t} \langle\, FS \,|\, e^{i\mathcal{H}_0 t} e^{-i(\mathcal{H}_0 - h_2)t} \,|\, FS \,\rangle \\ &= e^{i\left(E_f - \delta\right)t} \langle\, FS \,|\, e^{i\mathcal{H}_0 t} U^\dagger e^{-i\mathcal{H}_0 t} U \,|\, FS \,\rangle \\ &= e^{i\left(E_f - \delta\right)t} \langle\, FS \,|\, e^{-i\frac{v}{2\pi}\theta(t)} e^{i\frac{v}{2\pi}\theta(0)} \,|\, FS \,\rangle \end{aligned} \tag{7}$$

where $\theta(t)$ is given by,

$$\theta(t) = i \sum_{q>0} \sqrt{\frac{2\pi}{qL}} \left(b_q^\dagger e^{iqt} - b_q e^{-iqt} \right)$$

To solve for the expectation value in eqn(7) we use the result that $e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$ for $[A, B]$ commuting with A and B. We also note that in terms of the bosonic variables the Fermi sea is described by the absence of any excitation, i.e. $b_q \,|\, FS \,\rangle = 0$. Finally we get,

$$G(t) = e^{i\left(E_f - \delta\right)t} e^{-\left(\frac{v}{2\pi}\right)^2 \sum_{q>0} \left(\frac{2\pi}{qL}\right)\left(1 - e^{-iqt}\right)} \tag{8}$$

The q sum is converted into an integral, and for large t we have,

$$\int_0^\Lambda dq \frac{1 - e^{-iqt}}{q} \sim \int_{1/t}^\Lambda \frac{dq}{q} \sim \ln(\Lambda t)$$

so that we get the final result as,

$$G(t) = \frac{e^{i\left(E_f - \delta\right)t}}{(\Lambda t)^{\alpha^2}} \tag{9}$$

where $\alpha = \frac{v}{2\pi}$.

We note that in the absence of the interaction term the Green's function would be of the form $e^{iE_f t}$, i.e. it would represent undamped hole propagation. But the effect of the interaction is to give a power law decay of the Green's function (in Fermi liquid theory fermionic propagators have exponential decay), whose exponent depends on the interaction strength. This nonanalytic effect of the interaction cannot be obtained by an usual perturbative expansion in the interaction term. There is also a self energy correction to the f level due to interaction.

5. The Fourier transform of the Green's function is given by,

$$G(\omega) = \int_0^\infty dt G(t) e^{-i\omega t}$$

Fourier transform of a power law is a power law once again, and we have the result,

$$G(\omega) \sim \frac{1}{(\omega - E_f + \delta)^{1-\alpha^2}} \tag{10}$$

$G(\omega)$ is the density of states for the f electron. In the absence of interaction it was a delta function peaked around E_f. With the interaction, it has a power law divergence around $(E_f - \delta)$. The spread in the density of states is due to interaction with the conduction electrons.