Solution to Problem Set 3

We will consider a system of fermions interacting via a short range two body force with potential $V(r)$. The free part of the Hamiltonian is given by,

$$\mathcal{H}_0 = \sum_k \epsilon_k a_k^\dagger a_k \tag{1}$$

and the interacting part (which will be treated in perturbation theory) is given by,

$$\mathcal{H}_1 = \frac{1}{2}\int dr dr' V(r-r')\psi^\dagger(r)\psi(r)\psi^\dagger(r')\psi(r')$$

which in second quantized notation can be written as,

$$\mathcal{H}_1 = \frac{1}{2V}\sum_{k,k',q} V(q) a_{k+q}^\dagger a_k a_{k'-q}^\dagger a_{k'} \tag{2}$$

The Fourier transform of the potential is defined as,

$$V(r) = \frac{1}{V}\sum_q e^{iqr} V(q) \tag{3}$$

1. To first order in perturbation, self energy corrections are given by the Hartree and the Fock terms.

$$\begin{aligned}\Sigma^1(p,\omega) &= -i\frac{V(0)}{V}\sum^k \int \frac{d\Omega}{2\pi} G^0(k,\Omega)e^{i\Omega\eta} + i\frac{1}{V}\sum_k V(p-q)\int \frac{d\Omega}{2\pi}G^0(k,\Omega)e^{i\Omega\eta} \\ &= nV(0) - \frac{1}{V}\sum_{k\leq k_f} V(p-k) \end{aligned}\tag{4}$$

From the definition of effective mass,

$$\frac{p_f}{m^\star} = \frac{\partial \tilde{\epsilon_p}}{\partial p}|_{p=p_f}$$

we find that the Hartree term does not change the quasiparticle mass. If the interaction is momentum dependent, the Fock term renormalizes the mass to give,

$$\frac{p_f}{m^\star} = \frac{p_f}{m} - \frac{1}{V}\sum_{k\leq k_f}\frac{\partial V(p-k)}{\partial p}|_{p=p_f} \tag{5}$$

As an aside we will calculate the self energy using perturbation theory and Wick's theorem (i.e. not use diagrams and Green's function), and rederive eqn (4). To calculate energy upto first order in perturbation we need only the zeroth order wavefunctions. Then, the energy of the interacting Fermi sea is given by,

$$E_0 = \langle\, \phi_0 \,|\, \mathcal{H}_0 \,|\, \phi_0 \,\rangle + \langle\, \phi_0 \,|\, \mathcal{H}_1 \,|\, \phi_0 \,\rangle$$

and the energy of the one quasiparticle state is given by,

$$E_p = \langle\, \phi_0 \,|\, a_p \mathcal{H}_0 a_p^\dagger \,|\, \phi_0 \,\rangle + \langle\, \phi_0 \,|\, a_p \mathcal{H}_1 a_p^\dagger \,|\, \phi_0 \,\rangle$$

Then, the quasiparticle energy is given by,

$$\begin{aligned}
\tilde{\epsilon}_p &= E_p - E_0 \\
&= \langle\, \phi_0 \,|\, a_p \mathcal{H}_0 a_p^\dagger \,|\, \phi_0 \,\rangle - \langle\, \phi_0 \,|\, \mathcal{H}_0 \,|\, \phi_0 \,\rangle + \langle\, \phi_0 \,|\, a_p \mathcal{H}_1 a_p^\dagger \,|\, \phi_0 \,\rangle - \langle\, \phi_0 \,|\, \mathcal{H}_1 \,|\, \phi_0 \,\rangle \\
&= \epsilon_p + \Sigma_p^1
\end{aligned}$$

Thus the self energy of the quasiparticle is given by,

$$\begin{aligned}
\Sigma_p^1 &= \langle\, \phi_0 \,|\, a_p \mathcal{H}_1 a_p^\dagger \,|\, \phi_0 \,\rangle - \langle\, \phi_0 \,|\, \mathcal{H}_1 \,|\, \phi_0 \,\rangle \\
&= \frac{1}{2V} \sum_{k_1,k_2,q} V(q) \langle\, \phi_0 \,|\, a_p a_{k_1+q}^\dagger a_{k_1} a_{k_2-q}^\dagger a_{k_2} a_p^\dagger \,|\, \phi_0 \,\rangle - \frac{1}{2V} \sum_{k_1,k_2,q} V(q) \langle\, \phi_0 \,|\, a_{k_1+q}^\dagger a_{k_1} a_{k_2-q}^\dagger a_{k_2} \,|\, \phi_0 \,\rangle
\end{aligned}$$

Now, we use Wick's theorem to calculate the above matrix elements. We look at the first term above. There will be a contraction of the form $\langle\, \phi_0 \,|\, a_p a_p^\dagger \,|\, \phi_0 \,\rangle \langle\, \phi_0 \,|\, a_{k_1+q}^\dagger a_{k_1} a_{k_2-q}^\dagger a_{k_2} \,|\, \phi_0 \,\rangle$ which will cancel exactly with the second term in the above expression. This corresponds to correction to the Fermi sea, and in diagrammatic language, are the disconnected diagrams. The remaining two contractions will be $\langle\, a_{k_1+q}^\dagger a_{k_1} \,\rangle_0 \langle\, a_p a_{k_2-q}^\dagger \,\rangle_0 \langle\, a_{k_2} a_p^\dagger \,\rangle_0$, which gives the Hartree term, and $\langle\, a_p a_{k_1+q}^\dagger \,\rangle_0 \langle\, a_{k_1} a_{k_2-q}^\dagger \,\rangle_0 \langle\, a_{k_2} a_p^\dagger \,\rangle_0$, which gives the Fock term. Interchanging k_1 and k_2 give a factor of 2 which cancel with the 2 in the denominator. Finally we get,

$$\Sigma_p^1 = nV(0) + \frac{1}{V} \sum_{k \geq k_f} V(p-k)$$

which differ from equation (4) by a constant term. The origin of this diffrence can be traced back to the fact that we have used $a^\dagger a a^\dagger a$ in the form of the interaction, instead of $a^\dagger a^\dagger a a$ that is used in diagrammatic expansions.

2. The perturbed ground state $|\, \psi_0 \,\rangle$ is constructed from the unperturbed ground state $|\, \phi_0 \,\rangle$ by switching on the interaction adiabatically. This is mathematically expressed as $|\, \psi_0 \,\rangle = \hat{\Omega} \,|\, \phi_0 \,\rangle$, where $\hat{\Omega}$ is the evolution operator, and is given by,

$$\hat{\Omega} = Texp\{ \frac{1}{i} \int_{-\infty}^{\infty} d\tau \mathcal{H}_1^{int}(\tau) e^{\eta\tau} \} \quad , \quad \eta \to 0^+ \tag{6}$$

The time evolution of the interaction term in the interaction picture is given by,

$$\begin{aligned}
\mathcal{H}_1^{int}(t) &= e^{i\mathcal{H}_0 t} \mathcal{H}_1 e^{-i\mathcal{H}_0 t} \\
&= \frac{1}{2V} \sum_{k,k',p} V(p) a_{k+p}^\dagger a_k a_{k'-p}^\dagger a_{k'} e^{i\left(\epsilon_{k+p} + \epsilon_{k'-p} - \epsilon_k - \epsilon_{k'}\right)t}
\end{aligned} \tag{7}$$

Expanding the evolution operator to first order in interaction we get,

$$\begin{aligned}
\hat{\Omega} &= 1 + \frac{1}{i} \int_{-\infty}^{0} d\tau \mathcal{H}_1^{int}(\tau) e^{\eta\tau} \\
&= 1 + \frac{1}{2V} \sum_{k,k',p} \frac{V(p)}{(\epsilon_k + \epsilon_{k'} - \epsilon_{k+p} - \epsilon_{k'-p} + i\eta)} a_{k+p}^\dagger a_k a_{k'-p}^\dagger a_{k'}
\end{aligned} \tag{8}$$

Then the many body state $|\, q \,\rangle = \hat{\Omega} a_q^\dagger \,|\, \phi_0 \,\rangle$, to lowest order in interaction is given by,

$$|\, q \,\rangle = a_q^\dagger \,|\, \phi_0 \,\rangle + \frac{1}{2V} \sum_{k,k',p} \frac{V(p)}{(\epsilon_k + \epsilon_{k'} - \epsilon_{k+p} - \epsilon_{k'-p} + i\eta)} a_{k+p}^\dagger a_k a_{k'-p}^\dagger a_{k'} a_q^\dagger \,|\, \phi_0 \,\rangle \tag{9}$$

3. The charge density operator is given by,

$$\rho(r) = \frac{1}{V} \sum_q e^{iqr} \rho(q)$$

where,

$$\rho(q) = \sum_p a_{p+q}^\dagger a_p$$

The expectation of the charge density operator in the state $|\, k \,\rangle$ is given by,

$$\langle\, \rho(q) \,\rangle = \sum_p \langle\, k \,|\, a_{p+q}^\dagger a_p \,|\, k \,\rangle$$

Looking at the momentum of the states, it is easy to conclude that the above expectation values are zero except for $q = 0$. $\rho(q = 0)$ gives the total particle number, which is a conserved quantity. For a N particle ground state $|\, \phi_0 \,\rangle$, we have $\langle\, \rho(q = 0) \,\rangle = N + 1$. Then,

$$\langle\, \rho(r) \,\rangle = \frac{1}{V} \langle\, \rho(q = 0) \,\rangle = \frac{N + 1}{V} \tag{10}$$

Thus, the charge density is uniform and is delocalized over the entire system.

The current density operator is given by,

$$\bar{j}(r) = \frac{1}{V} \sum_q e^{iqr} \bar{j}(q)$$

where,

$$\bar{j}(q) = \sum_p \frac{1}{m} (\bar{p} - \frac{\bar{q}}{2}) a_{p+q}^\dagger a_p$$

By the same argument as above the average quantity $\langle\, k \,|\, \bar{j}(q) \,|\, k \,\rangle$ is zero except for $q = 0$. For a translationally invariant system, total momentum is a conserved quatity. Particularly, $\left[\hat{\Omega}, \sum_p \bar{p} a_p^\dagger a_p\right] = 0$, so that $\langle\, k \,|\, \bar{j}(q = 0) \,|\, k \,\rangle = \frac{\bar{k}}{m}$. It is to be noted that what enters in the expression for current density is the bare particle mass and not the renormalized quasiparticle mass, and that this is a consequence of translational invariance. Thus,

$$\langle\, \bar{j}(r) \,\rangle = \frac{1}{V} \left(\frac{\bar{k}}{m} \right) \tag{11}$$

The current density due to the quasiparticle excitation is uniformly spread.

4. We consider the wave-packet state $|\, x_0 \,\rangle = \sum_q A_q \,|\, q \,\rangle$, where $A_q = e^{-\alpha(q-q_0)^2} e^{-iqx_0}$, constructed out of single particle excitations. We expect this state to behave as an excitation that is localized around x_0. To find whether indeed that is the case, we look at the average charge density $\langle\, \rho_q \,\rangle$ in this state. We have,

$$\langle\, x_0 \,|\, \rho_q \,|\, x_0 \,\rangle = \sum_{k,p} A_k A^\star_{k+q} \langle\, k+q \,|\, a^\dagger_{p+q} a_p \,|\, k \,\rangle \tag{12}$$

As before the $q = 0$ component measures the total particle number, which is $N + 1$ for a N particle ground state. Thus,

$$\langle\, x_0 \,|\, \rho_0 \,|\, x_0 \,\rangle = (N+1) \sum_k A_k A^\star_k \tag{13}$$

For $q \neq 0$ components we expand the many particle wavefunction $|\, k \,\rangle$ in perturbation theory. Then the zeroth order contribution to the matrix element on the right hand side of eqn(12) is $\langle\, \phi_0 \,|\, a_{k+q} a^\dagger_{p+q} a_p a^\dagger_k \,|\, \phi_0 \,\rangle = \delta_{p,k}$. From the result of part (2) we get the first order contribution as,

$$\frac{1}{V} \sum_{k_1,k_2,k_3} \frac{V(k_3)}{\epsilon_{k_1} + \epsilon_{k_2} - \epsilon_{k_1+k_3} - \epsilon_{k_2-k_3} + i\eta} \langle\, \phi_0 \,|\, a_{k+q} \left[a^\dagger_{p+q} a_p, a^\dagger_{k_1+k_3} a_{k_1} a^\dagger_{k_2-k_3} a_{k_2} \right] a^\dagger_k \,|\, \phi_0 \,\rangle$$

Using Wick's theorem to simplify the above expression, we find that there are only two non-zero terms, and we get,

$$\langle\, \rho_q \,\rangle = \sum_k A_k A^\star_{k+q} \left\{ 1 - \frac{1}{V} \sum_p \frac{V(q) - V(k-p)}{q \cdot (v_p - v_k)} \left(f^0_p - f^0_{p+q} \right) \right\} \tag{14}$$

where f^0 is equilibrium Fermi distribution. The most important observation is that the second term in the right hand side above has a nonzero limit as $q \to 0$. Thus ρ_q is discontinuous at $q = 0$. The total charge that is localized is given by,

$$\lim_{q \to 0} \rho_q = \sum_k A_k A^\star_k \left\{ 1 - \frac{1}{V} \sum_p \frac{V(0) - V(k-p)}{q \cdot (v_p - v_k)} \delta(\epsilon_p)(v_p \cdot q) \right\} \tag{15}$$

Since $\lim_{q \to 0} \rho_q \neq \delta\rho_{q=0} = \sum_k A_k A^\star_k$, the entire charge of the excitation is not localized. The delocalized part is given by the second term on the right hand side of eqn(15). Thus we find that it is not possible to construct entirely localized objects from single particle excitations. And the reason for this is that in systems with Fermi surfaces there are large number of excitations with arbirary small energies. These low energy processes are responsible for the delocalization of the charge.

The average current density in the state $|\, x_0 \,\rangle$ is given by,

$$\langle\, x_0 \,|\, \bar{j}_q \,|\, x_0 \,\rangle = \sum_k A_k A^\star_{k+q} \sum_p \frac{1}{m} \left(\bar{p} - \frac{\bar{q}}{2} \right) \langle\, k+q \,|\, a^\dagger_{p+q} a_p \,|\, k \,\rangle \tag{16}$$

The $q = 0$ component gives the total current in the system. Since total momentum is conserved, we find,

$$\langle \bar{j}_0 \rangle = \sum_k A_k A_k^\star \left(\frac{\bar{k}}{m}\right) = \frac{\bar{q}_0}{m} \sum_k A_k A_k^\star \tag{17}$$

while for $q \neq 0$ components we have

$$\langle \bar{j}_q \rangle = \sum_k A_k A_{k+q}^\star \sum_p \frac{1}{m}\left(\bar{p} - \frac{\bar{q}}{2}\right) \left\{\delta_{p,k} - \frac{1}{V}\frac{V(q) - V(k-p)}{q \cdot (v_p - v_k)} \left(f_p^0 - f_{p+q}^0\right)\right\} \tag{18}$$

As before, $\lim_{q\to 0} \langle \bar{j}_q \rangle \neq \langle \bar{j}_0 \rangle$, and we find that part of the current is delocalized.

5. To find an expression for conductivity, we look at the Boltzman equation,

$$\frac{\partial n(k,r,t)}{\partial t} + \frac{\partial n}{\partial r}\frac{\partial \epsilon}{\partial k} - \frac{\partial n}{\partial k}\frac{\partial \epsilon}{\partial r} = 0$$

Here we are disregarding the effect of collision that appears in the right hand side of the Boltzman equation. The collision frquency ν goes as T^2, and so at low enough temperature the above approximation is valid. Let $n(k,r,t) = n_0(k) + \delta n(k,r,t)$, where n_0 is the equilibrium Fermi distribution function. Linearizing the Boltzman equation we get,

$$\frac{\partial \delta n}{\partial t} + \frac{\partial \delta n}{\partial r}\frac{\partial \epsilon_k^0}{\partial k} - \frac{\partial n_0}{\partial k}\frac{\partial \epsilon}{\partial r} = 0$$

Let there be an external electric field $\bar{E} = \bar{E}_0 e^{iqr - i\omega t}$ where $\omega \gg \nu$. Then,

$$\epsilon_k = \epsilon_k^0 + \sum_{k'} f(k,k')\delta n(k') - eEr$$

Let the shift from the equilibrium be $\delta n(k,r,t) = \delta n(k) e^{iqr - i\omega t}$. Then, from the Boltzman equation we get,

$$(q \cdot v_k - \omega)\,\delta n(k) + q \cdot v_k \delta(\epsilon_k - \mu) \sum_{k'} f(k,k')\delta n(k') + ieE_0 \cdot v_k \delta(\epsilon_k - \mu) = 0$$

For $q = 0$, i.e. a spatially uniform external field we get,

$$\delta n(k) = \frac{ieE_0 \cdot v_k}{\omega}\delta(\epsilon_k - \mu) \tag{19}$$

The current density is given by $\bar{j} = e\sum_k \delta n(k)\bar{j}_k$, where for a translationally invariant system we have already found that $\bar{j}_k = \frac{\bar{k}}{m}$. Then,

$$\begin{aligned} \bar{j} &= \frac{ie^2}{\omega}\sum_k E_0 \cdot v_k \frac{\bar{k}}{m}\delta(\epsilon_k - \mu) \\ &= \frac{iNe^2}{m\omega}\bar{E}_0 \end{aligned} \tag{20}$$

so that $\sigma = \frac{iNe^2}{m\omega}$. Thus the constant C has the value,

$$C = \frac{-Ne^2}{m} \tag{21}$$

Due to Galilean invariance the quasiparticle current density is $\frac{\bar{k}}{m}$, and so the expression for conductivity has a factor of m and not $m^\star$. Thus there is no correction from $V(r)$.