Solution to Problem Set 1

For a system of non-interacting fermions described by the Hamiltonian,

$$\mathcal{H}_0 = \sum_k \epsilon_k a_k^\dagger a_k$$

we want to calculate Green's functions with respect to a state $|\,\psi\,\rangle = \prod_i a_k^\dagger\,|\,0\,\rangle$, that is characterized by a distribution function

$$n_k = \langle\,\psi\,|\,a_k^\dagger a_k\,|\,\psi\,\rangle$$

The Fourier transform of a field operator $\psi(x,t)$ is defined as,

$$\psi(x,t) = \int \frac{d^3k}{(2\pi)^3} e^{ikx}\psi(k,t)$$

and the time evolution of the Heisenberg operator $\psi(k,t)$ is,

$$\begin{aligned}\psi(k,t) &= e^{i\mathcal{H}_0 t} a_k e^{-i\mathcal{H}_o t} \\ &= e^{-i\epsilon_k t} a_k\end{aligned}$$

Note that time evolution giving just a phase factor is a simplification only for a noninteracting system (all the fuss about interacting systems is due to this unfortunate fact)

1. $G^>(1,2)$ is defined as $-i\langle\,\psi(x_1,t_1)\psi^\dagger(x_2,t_2)\,\rangle$. Performing the Fourier transform and the time evolution as defined above, we get,

$$G^>(1,2) = -i\int \frac{d^3k_1}{(2\pi)^3}\int \frac{d^3k_2}{(2\pi)^3} e^{ik_1x_1 - ik_2x_2} e^{-i\epsilon_{k_1}t_1 + i\epsilon_{k_2}t_2}\langle\, a_{k_1}a_{k_2}^\dagger\,\rangle$$

From the anticommutation of the second quantized operators we get,

$$\langle\, a_{k_1}a_{k_2}^\dagger\,\rangle = (2\pi)^3\delta(k_1 - k_2)(1 - n_{k_1})$$

so the above expression simplifies to

$$G^>(1,2) = -i\int \frac{d^3k}{(2\pi)^3} e^{ik(x_1-x_2)} e^{-i\epsilon_k(t_1-t_2)}(1 - n_k)$$

Since the system is translationally invariant, as expected, $G^>$ is a function of $(x_1 - x_2)$ and $(t_1 - t_2)$. Let $(x_1 - x_2) \equiv x$ and $(t_1 - t_2) \equiv t$. Then the Fourier transform of $G^>(x,t)$ is

$$G^>(k,\omega)) = 2\pi i\delta(\omega - \epsilon_k)(n_k - 1) \tag{1}$$

2. Evaluation of $G^<(1,2)$ is similar and we get

$$\begin{aligned} G^<(1,2) &= -i\langle\, \psi^\dagger(x_2,t_2)\psi(x_1,t_1)\,\rangle \\ &= i\int \frac{d^3k}{(2\pi)^3} e^{ikx-i\epsilon_k t} n_k \end{aligned}$$

whose Fourier transform is

$$G^<(k,\omega)) = 2\pi i\delta(\omega-\epsilon_k)n_k \tag{2}$$

3. The time ordered Green's function is defined as

$$\begin{aligned} G^t(1,2) &= -i\langle\, T(\psi(x_1,t_1)\psi^\dagger(x_2,t_2))\,\rangle \\ &= \theta(t_1-t_2)G^>(1,2)+\theta(t_2-t_1)G^<(1,2) \end{aligned}$$

or

$$G^t(x,t) = -i\int \frac{d^3k}{(2\pi)^3} e^{ikx-i\epsilon_k t}\theta(t)(1-n_k) + i\int \frac{d^3k}{(2\pi)^3} e^{ikx-i\epsilon_k t}\theta(-t)n_k$$

Since

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega-\epsilon_k+i\eta} = -ie^{-i\epsilon_k t}\theta(t)$$

and

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega-\epsilon_k-i\eta} = ie^{-i\epsilon_k t}\theta(-t)$$

the Fourier transform of the time ordered Green's function is

$$G^t(k,\omega) = \frac{1-n_k}{\omega-\epsilon_k+i\eta} + \frac{n_k}{\omega-\epsilon_k-i\eta} \tag{3}$$

The first term correspond to particle propagation in positive (future) time while the second term describes hole propagation in negative (past) time.

4. The anti time ordered Green's function, defined as

$$\begin{aligned} G^{\tilde t}(1,2) &= -i\langle\, \tilde T(\psi(x_1,t_1)\psi^\dagger(x_2,t_2))\,\rangle \\ &= \theta(t_2-t_1)G^>(1,2)+\theta(t_1-t_2)G^<(1,2) \end{aligned}$$

is evaluated similarly, and it's Fourier transform is

$$G^{\tilde t}(k,\omega) = -\left[\frac{1-n_k}{\omega-\epsilon_k-i\eta} + \frac{n_k}{\omega-\epsilon_k+i\eta}\right] \tag{4}$$

5. The retarded Green's function, defined as

$$\begin{aligned} G^R(1,2) &= -i\theta(t_1-t_2)\langle\, \{\psi(x_1,t_1),\psi^\dagger(x_2,t_2)\}\,\rangle \\ &= \theta(t_1-t_2)\left[G^>(1,2)-G^<(1,2)\right] \\ &= -i\int \frac{d^3k}{(2\pi)^3} e^{ikx-i\epsilon_k t}\theta(t) \end{aligned}$$

and it's Fourier transform is

$$G^R(k,\omega) = \frac{1}{\omega - \epsilon_k + i\eta} \tag{5}$$

6. Similarly the advanced Green's function, defined as

$$G^A(1,2) = i\theta(t_2 - t_1)\langle\, \{\psi(x_1,t_1), \psi^\dagger(x_2,t_2)\}\,\rangle$$

has the Fourier transform

$$G^A(k,\omega) = \frac{1}{\omega - \epsilon_k - i\eta} \tag{6}$$

The retarded and the advanced Green's functions do not carry any information about the distribution function.

7.

$$\begin{aligned} G^K(1,2) &= -i\langle\, [\psi(x_1,t_1), \psi^\dagger(x_2,t_2)]\,\rangle \\ &= G^>(1,2) + G^<(1,2) \end{aligned}$$

and in the Fourier space this has the form

$$G^K(k,\omega) = 2\pi i\delta(\omega - \epsilon_k)(2n_k - 1) \tag{7}$$

8. The Matsubara Green's function is defined as $g(\tau,x) = -\langle\, T_\tau \psi(\tau,x)\psi^\dagger(0,0)\,\rangle$, where τ is imaginary time, and T_τ orders imaginary time. The (imaginary) time development of $\psi(\tau,k)$ is given by

$$\begin{aligned} \psi(\tau,k) &= e^{\tau K_0} a_k e^{-\tau K_0} \\ &= e^{-\tau\xi_k} a_k \end{aligned}$$

where $K_0 = \mathcal{H}_0 - \mu N$, and $\xi_k = \epsilon_k - \mu$. Then,

$$\begin{aligned} g(\tau,x) &= -\int \frac{d^3k}{(2\pi)^3} e^{ikx} e^{-\tau\xi_k} \{\theta(\tau)\langle\, a_k a_k^\dagger\,\rangle - \theta(-\tau)\langle\, a^{\dagger k} a_k\,\rangle\} \\ &= -\int \frac{d^3k}{(2\pi)^3} e^{ikx} e^{-\tau\xi_k} \{\theta(\tau) - n_F(\xi_k)\} \end{aligned}$$

where $n_F(\xi_k) = \frac{1}{e^{\beta\xi_k}+1}$, is the fermi function. For a fermionic system, $g(\tau)$ satisfies antiperiodic boundary condition, so that, $g(\tau+\beta) = -g(\tau)$, where β is inverse temperature in energy units. Therefore the Fourier transform of $g(\tau)$ (in imaginary frequency) is defined as

$$g(i\omega_n,k) = \int_0^\beta d\tau g(\tau,k) e^{i\tau\omega_n}$$

where $\omega_n = \frac{(2n+1)\pi}{\beta}$, are the fermionic Matsubara frequencies. The result of the Fourier transform is,

$$g(i\omega_n,k) = \frac{1}{i\omega_n - \xi_k} \tag{8}$$

$g(i\omega_n, k)$ has no explicit dependence on the distribution function $n_F(\xi_k)$. But while carrying out the discrete sum over fermionic frequencies the distribution function reappears, since the discrete frequencies are at the poles of the fermi distribution function. Analytic continuation to real frequencies ($i\omega_n \longrightarrow \omega + \pm i\eta$) give the retarded and advanced Green's functions respectively, which we have already evaluated. Note that the corresponding expressions match.