Solution to Problem Set 4

We are examining a system of free fermions characterized by the Hamiltonian,

$$\mathcal{H}_0 = \sum_{k,\sigma} \epsilon_k c^\dagger_{k,\sigma} c_{k,\sigma},$$

where $\epsilon_k = k^2/2m$. The ground state of such a system is a filled Fermi sea. It can be represented as

$$|\psi_0\rangle = \prod_{k<k_F} c^\dagger_{k,\uparrow} c^\dagger_{k,\downarrow}.$$

Zero Temperature Calculation

We have to evaluate the correlation function,

$$\begin{aligned} G_\rho(t,q) &= (-i)\langle T\left(\rho_q(t)\rho_{-q}(0)\right)\rangle \\ &= (-i)\{\theta(t)\langle \rho_q(t)\rho_{-q}(0)\rangle + \theta(-t)\langle \rho_{-q}(0)\rho_q(t)\rangle\}, \end{aligned} \quad (1)$$

where $\rho_q \equiv \sum_{k,\sigma} c^\dagger_{k+q,\sigma} c_{k,\sigma}$. $\langle \hat{O}\rangle$ denotes the ground state average $\langle \psi_0|\hat{O}|\psi_0\rangle$, since we are evaluating matrix elements at zero temperature. For a free theory the time evolution of the Heisenberg operators is simple. It is easy to verify that

$$e^{i\mathcal{H}_o t} c^\dagger_k e^{-i\mathcal{H}_o t} = e^{i\epsilon_k t} c^\dagger_k,$$

and

$$e^{i\mathcal{H}_o t} c_k e^{-i\mathcal{H}_o t} = e^{-i\epsilon_k t} c_k.$$

Then,

$$\begin{aligned} \rho_q(t) &= e^{i\mathcal{H}_o t}\rho_q(0)e^{-i\mathcal{H}_o t} \\ &= \sum_{k,\sigma} e^{i(\epsilon_{k+q}-\epsilon_k)t} c^\dagger_{k+q,\sigma} c_{k,\sigma}. \end{aligned} \quad (2)$$

From equations (1) and (2) we get,

$$\begin{aligned} G_\rho(t,q) = (-i) \sum_{k,k',\sigma,\sigma'} e^{i(\epsilon_{k+q}-\epsilon_k)t} &\Big\{\theta(t)\langle c^\dagger_{k+q,\sigma} c_{k,\sigma} c^\dagger_{k'-q,\sigma'} c_{k',\sigma'}\rangle \\ &+\theta(-t)\langle c^\dagger_{k'-q,\sigma'} c_{k',\sigma'} c^\dagger_{k+q,\sigma} c_{k,\sigma}\rangle\Big\}. \end{aligned} \quad (3)$$

To evaluate the matrix elements in the above expression we have to remember two things. First, $\langle c^\dagger_{k,\sigma} c_{k',\sigma'}\rangle = \delta_{k,k'}\delta_{\sigma,\sigma'} n_k$, since we can create a hole only from within the Fermi sea, and $\langle c_{k,\sigma} c^\dagger_{k',\sigma'}\rangle = \delta_{k,k'}\delta_{\sigma,\sigma'}(1-n_k)$, since we can create a particle only outside the Fermi sea. Here n_k (zero temperature Fermi function) is the particle occupation number of the state k. Second, an expression like $\langle c^\dagger_1 c_2 c^\dagger_3 c_4\rangle$ is non-zero only when whatever particle is destroyed (or created) from the Fermi sea is created (or destroyed) back. Thus the index in c_4 must match that of either $c_1^\dagger$ or $c_3^\dagger$. In the jargon this is usually known as contarction. With these two observations it is easy to see that,

$$\begin{aligned}
\langle\, c^\dagger_{k+q,\sigma} c_{k,\sigma} c^\dagger_{k'-q,\sigma'} c_{k',\sigma'} \,\rangle &= \langle\, c^\dagger_{k+q,\sigma} c_{k,\sigma} \,\rangle \langle\, c^\dagger_{k'-q,\sigma'} c_{k',\sigma'} \,\rangle + \langle\, c^\dagger_{k+q,\sigma} c_{k',\sigma'} \,\rangle \langle\, c_{k,\sigma} c^\dagger_{k'-q,\sigma'} \,\rangle \\
&= \delta_{q,0}\; n_{k,\sigma}\; n_{k',\sigma'} + \delta_{\sigma,\sigma'}\; \delta_{k',k+q}\; n_{k+q,\sigma}(1 - n_{k,\sigma}).
\end{aligned}$$

Similarly we get,

$$\langle\, c^\dagger_{k'-q,\sigma'} c_{k',\sigma'} c^\dagger_{k+q,\sigma} c_{k,\sigma} \,\rangle = \delta_{q,0}\; n_{k,\sigma}\; n_{k',\sigma'} + \delta_{\sigma,\sigma'}\; \delta_{k',k+q}\; n_{k,\sigma}(1 - n_{k+q,\sigma})$$

Then, from equation (3) we get,

$$G_\rho(t,q) = (-i)N^2\delta_{q,0} + (-2i)\sum_k e^{i(\epsilon_{k+q}-\epsilon_k)t}\left\{\theta(t) n_{k+q}(1-n_k) + \theta(-t) n_k(1-n_{k+q})\right\}, \qquad (4)$$

where the factor of two in the second term is from a sum over spin indices. The first term is due to the constant density of the Fermi sea. For the purpose of studying density fluctuations the first term is irrelevant, and in further analysis we drop it.

Next we represent the theta function by the identity,

$$\theta(t) = -\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i}\, \frac{e^{-i\omega' t}}{\omega' + i\eta}, \qquad (5)$$

because only for $t > 0$, we can close the contour in the lower half plane ($Im\omega' < 0$), and pick the pole at $\omega' = -i\eta$. Similarly,

$$\theta(-t) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i}\, \frac{e^{-i\omega' t}}{\omega' - i\eta}. \qquad (6)$$

Using these identities we can write,

$$G_\rho(t,q) = (2i)\sum_k e^{i(\epsilon_{k+q}-\epsilon_k)t} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i} e^{-i\omega' t}\left\{ \frac{n_{k+q}(1-n_k)}{\omega' + i\eta} - \frac{n_k(1-n_{k+q})}{\omega' - i\eta} \right\}.$$

Now we can write the Fourier transform of the density-density correlator as,

$$\begin{aligned}
G_\rho(\omega,q) &= \int_{-\infty}^{\infty} dt e^{i\omega t} G_\rho(t,q) \\
&= 2\sum_k \left\{ \frac{n_{k+q}(1-n_k)}{\omega + \epsilon_{k+q} - \epsilon_k + i\eta} - \frac{n_k(1-n_{k+q})}{\omega + \epsilon_{k+q} - \epsilon_k - i\eta} \right\} \qquad (7)
\end{aligned}$$

The poles in the above expression are the energy for particle-hole excitations of the Fermi sea. Hence, $G_\rho(\omega, q)$ is also called the particle-hole propagator. These exciations are possible modes by which the system can absorb energy and go to an excited state from its ground state.

The above expression can be simplified further if in the first term we replace $\vec{k}$ by $-\vec{k} - \vec{q}$. Since $\epsilon_{\vec{k}} = \epsilon_{-\vec{k}}$, we get,

$$G_\rho(\omega,q) = 2\sum_k n_k(1-n_{k+q})\left\{ \frac{1}{\omega + \epsilon_k - \epsilon_{k+q} + i\eta} - \frac{1}{\omega + \epsilon_{k+q} - \epsilon_k - i\eta} \right\} \qquad (8)$$

Next we have to evaluate the correlation function $G_p(t,q) = (-i)\langle\, T\left(\Delta_q(t)\Delta_q^\dagger(0)\right)\,\rangle$, where $\Delta_q^\dagger \equiv \sum_k c^\dagger_{-k+q,\uparrow} c^\dagger_{k,\downarrow}$. Proceeding as before we get,

$$\begin{aligned} G_p(t,q) &= (-i)\left\{\theta(t)\langle\, \Delta_q(t)\Delta_q^\dagger(0)\,\rangle + \theta(-t)\langle\, \Delta_q^\dagger(0)\Delta_q(t)\,\rangle\right\} \\ &= (-i)\sum_{k,k'} e^{-i(\epsilon_k+\epsilon_{-k+q})t}\left\{\theta(t)\langle\, c_{k,\downarrow}c_{-k+q,\uparrow}c^\dagger_{-k'+q,\uparrow}c^\dagger_{k',\downarrow}\,\rangle\right. \\ &\quad \left. +\theta(-t)\langle\, c^\dagger_{-k'+q,\uparrow}c^\dagger_{k',\downarrow}c_{k,\downarrow}c_{-k+q,\uparrow}\,\rangle\right\}. \end{aligned} \tag{9}$$

Since,

$$\langle\, c_{k,\downarrow}c_{-k+q,\uparrow}c^\dagger_{-k'+q,\uparrow}c^\dagger_{k',\downarrow}\,\rangle = \delta_{k,k'}(1-n_{k,\downarrow})(1-n_{-k+q,\uparrow})$$

and

$$\langle\, c^\dagger_{-k'+q,\uparrow}c^\dagger_{k',\downarrow}c_{k,\downarrow}c_{-k+q,\uparrow}\,\rangle = \delta_{k,k'}n_{k,\downarrow}n_{-k+q,\uparrow},$$

we get,

$$G_p(t,q) = (-i)\sum_k e^{-i(\epsilon_k+\epsilon_{-k+q})t}\left\{\theta(t)(1-n_{k,\downarrow})(1-n_{-k+q,\uparrow}) + \theta(-t)n_{k,\downarrow}n_{-k+q,\uparrow}\right\} \tag{10}$$

We can now evaluate the Fourier transform of the correlator, as we did before. The final result is,

$$G_p(t,q) = \sum_k\left\{\frac{(1-n_{k,\downarrow})(1-n_{-k+q,\uparrow})}{\omega-\epsilon_k-\epsilon_{-k+q}+i\eta} - \frac{n_{k,\downarrow}n_{-k+q,\uparrow}}{\omega-\epsilon_k-\epsilon_{-k+q}-i\eta}\right\} \tag{11}$$

The poles of this function are the energies required to create particle-particle or hole-hole excitations. The first term in the above expression is the retarded part which gives two particle excitations, and the second term is the advanced part that gives two hole excitations.

Finite Temperature Calculation

We will now evaluate the two correlation functions at finite temperature and contrast the calculation with the zero temperature case. The density-density correlation function is defined as,

$$\begin{aligned} G_\rho(\tau,q) &= -\frac{1}{Z}\mathrm{Tr} e^{-\beta(\mathcal{H}_o-\mu N)}T_\tau\rho_q(\tau)\rho_{-q}(0) \\ &= -\langle\, T_\tau\rho_q(\tau)\rho_{-q}(0)\,\rangle, \end{aligned}$$

where now $\langle\, \hat{O}\,\rangle = \frac{1}{Z}\mathrm{Tr}(e^{-\beta(\mathcal{H}_o-\mu N)}\hat{O})$. τ is now imaginary time defined as $\tau = it$. The Heisenberg operators now evolve in imaginary time, and for a system of free particles it is easy to show that

$$e^{(\mathcal{H}_0-\mu N)\tau}c_k^\dagger e^{-(\mathcal{H}_0-\mu N)\tau} = e^{(\epsilon_k-\mu)\tau}c_k^\dagger,$$

and

$$e^{(\mathcal{H}_0-\mu N)\tau}c_k e^{-(\mathcal{H}_0-\mu N)\tau} = e^{-(\epsilon_k-\mu)\tau}c_k.$$

Then,

$$\rho_q(\tau) = \sum_{k,\sigma} e^{(\epsilon_{k+q}-\epsilon_k)\tau}c^\dagger_{k+q,\sigma}c_{k,\sigma}.$$

Thus,

$$G_\rho(\tau,q) = -\sum_{k,k',\sigma,\sigma'} e^{(\epsilon_{k+q}-\epsilon_k)\tau}\left\{\theta(\tau)\langle\, c^\dagger_{k+q,\sigma}c_{k,\sigma}c^\dagger_{k'-q,\sigma'}c_{k',\sigma'}\,\rangle + \theta(-\tau)\langle\, c^\dagger_{k'-q,\sigma'}c_{k',\sigma'}c^\dagger_{k+q,\sigma}c_{k,\sigma}\,\rangle\right\} \tag{12}$$

The evaluation of the matrix elements is similar to the zero temperature case. The only difference is that now $\langle\, c^\dagger_{k,\sigma}c_{k',\sigma'}\,\rangle = \delta_{k,k'}\delta_{\sigma,\sigma'}f_k$, where f_k is the Fermi function that gives average particle occupation in the state k at finite temperature. Similarly, $\langle\, c_{k,\sigma}c^\dagger_{k',\sigma'}\,\rangle = \delta_{k,k'}\delta_{\sigma,\sigma'}(1-f_k)$. With this observation we can write,

$$\langle\, c^\dagger_{k+q,\sigma}c_{k,\sigma}c^\dagger_{k'-q,\sigma'}c_{k',\sigma'}\,\rangle = \delta_{q,0}f_{k,\sigma}f_{k',\sigma'} + \delta_{k',k+q}\delta_{\sigma,\sigma'}f_{k+q,\sigma}(1-f_{k,\sigma}),$$

and

$$\langle\, c^\dagger_{k'-q,\sigma'}c_{k',\sigma'}c^\dagger_{k+q,\sigma}c_{k,\sigma}\,\rangle = \delta_{q,0}f_{k,\sigma}f_{k',\sigma'} + \delta_{k',k+q}\delta_{\sigma,\sigma'}f_{k,\sigma}(1-f_{k+q,\sigma}).$$

Ignoring the term from the constant average density, we get,

$$G_\rho(\tau,q) = -2\sum_k e^{(\epsilon_{k+q}-\epsilon_k)\tau}\left\{\theta(\tau)f_{k+q}(1-f_k) + \theta(-\tau)f_k(1-f_{k+q})\right\}. \tag{13}$$

The next thing to note is that $G_\rho(\tau)$ is periodic with period β, as we expect bosonic propagators to be. To verify this we will use the identity

$$\frac{f_k}{1-f_k} = e^{-\beta\epsilon_k}. \tag{14}$$

Then, for $\tau < 0$, we have,

$$\begin{aligned} G_\rho(\tau) &= -2\sum_k e^{(\epsilon_{k+q}-\epsilon_k)\tau}f_k(1-f_{k+q}) \\ &= -2\sum_k e^{(\epsilon_{k+q}-\epsilon_k)(\tau+\beta)}f_{k+q}(1-f_k) \\ &= G_\rho(\tau+\beta), \end{aligned}$$

for $\tau > -\beta$. Thus, we have proved that over the range $-\beta < \tau < \beta$, $G_\rho(\tau,q)$ is periodic. The Fourier transform of $G_\rho(\tau)$ in imaginary frequency is defined as,

$$G_\rho(\tau) = \frac{1}{\beta}\sum_n e^{-i\omega_n\tau}G_\rho(i\omega_n).$$

To satisfy the periodic condition, one needs $e^{-i\omega_n\beta} = 1$, or $\omega_n = 2\pi n/\beta$, i.e. bosonic Matsubara frequencies. The evaluation of $G_\rho(i\omega_n,q)$ is now straightforward, and we get,

$$\begin{aligned} G_\rho(i\omega_n,q) &= \int_0^\beta d\tau e^{i\omega_n\tau}G_\rho(\tau,q) \\ &= (-2)\sum_k\int_0^\beta d\tau e^{(i\omega_n+\epsilon_{k+q}-\epsilon_k)\tau}f_{k+q}(1-f_k) \\ &= (-2)\sum_k\frac{f_{k+q}(1-f_k)}{i\omega_n+\epsilon_{k+q}-\epsilon_k}\left(e^{\beta(\epsilon_{k+q}-\epsilon_k)}-1\right) \\ &= 2\sum_k\frac{f_{k+q}-f_k}{i\omega_n+\epsilon_{k+q}-\epsilon_k} \end{aligned} \tag{15}$$

The technology of evaluating the particle-particle (or hole-hole) correlation function is along similar lines, and we get,

$$\begin{aligned} G_p(\tau,q) &= -\frac{1}{Z}\mathrm{Tr}e^{-\beta(\mathcal{H}_o-\mu N)}T_\tau\left(\Delta_q(\tau)\Delta_q^\dagger(0)\right) \\ &= -\sum_{k,k'} e^{-(\epsilon_k+\epsilon_{-k+q}-2\mu)\tau}\left\{\theta(\tau)\langle\, c_{k,\downarrow}c_{-k+q,\uparrow}c^\dagger_{-k'+q,\uparrow}c^\dagger_{k',\downarrow}\,\rangle+\theta(-\tau)\langle\, c^\dagger_{-k'+q,\uparrow}c^\dagger_{k',\downarrow}c_{k,\downarrow}c_{-k+q,\uparrow}\,\rangle\right\} \\ &= -\sum_k e^{-(\epsilon_k+\epsilon_{-k+q}-2\mu)\tau}\left\{\theta(\tau)(1-f_{k,\downarrow})(1-f_{-k+q,\uparrow})+\theta(-\tau)f_{k,\downarrow}f_{-k+q,\uparrow}\right\} \qquad (16)\end{aligned}$$

As in the previous case, we can now show that $G_p(\tau,q)$ is periodic over the range $-\beta<\tau<\beta$, so that the Fourier modes in imaginary frequencies are only the bosonic Matsubara frequencies. Finally, the result of the Fourier transform is,

$$\begin{aligned} G_p(i\omega_n,q) &= \int_0^\beta d\tau e^{i\omega_n\tau}G_p(\tau,q) \\ &= -\sum_k(1-f_{k,\downarrow})(1-f_{-k+q,\uparrow})\int_0^\beta d\tau e^{(i\omega_n-\epsilon_k-\epsilon_{-k+q}+2\mu)\tau} \\ &= -\sum_k\frac{(1-f_{k,\downarrow})(1-f_{-k+q,\uparrow})}{i\omega_n-\epsilon_k-\epsilon_{-k+q}+2\mu}\left(e^{-\beta(\epsilon_k+\epsilon_{-k+q}-2\mu)}-1\right) \\ &= \sum_k\frac{1-f_{k,\downarrow}-f_{-k+q,\uparrow}}{i\omega_n-\epsilon_k-\epsilon_{-k+q}+2\mu} \qquad (17)\end{aligned}$$

Evaluation of Integrals at Zero temperature

1. **$G_\rho(\omega,q)$ in 3 dimensions**

   From equation (8) we note that $G_\rho(\omega,q)$ is even in ω. So we can restrict our discussion to $\omega>0$. Then,

$$G_\rho^R(\omega,q)=2\sum_k n_k(1-n_{k+q})\left\{P\frac{1}{\omega+\epsilon_k-\epsilon_{k+q}}-P\frac{1}{\omega+\epsilon_{k+q}-\epsilon_k}\right\}.$$

   Since under the transformation $\vec{k}\to-\vec{k}-\vec{q}$, the term $n_k n_{k+q}\left(1/(\omega+\epsilon_k-\epsilon_{k+q})-1/(\omega+\epsilon_{k+q}-\epsilon_k)\right)$ changes sign, therefore it must be zero. Then,

$$G_\rho^R(\omega,q)=2\sum_k n_k\left\{P\frac{1}{\omega+\epsilon_k-\epsilon_{k+q}}-P\frac{1}{\omega+\epsilon_{k+q}-\epsilon_k}\right\}. \qquad (18)$$

   In the continuum limit this can be written as,

$$G_\rho^R(\omega,q)=\frac{4\pi}{(2\pi)^3}\int_0^{k_F}dkk^2\int_{-1}^1 dz\left\{\frac{1}{\omega-\epsilon_q-\frac{kqz}{m}}-\frac{1}{\omega+\epsilon_q+\frac{kqz}{m}}\right\}.$$

We will introduce the dimensionless parameters $q' = q/k_F, k' = k/k_F$ and $\omega' = \omega m/k_F^2$. Dropping the primes we can rewrite the above expression in terms of dimensionless parameters as,

$$G^R_\rho(\omega, q) = \frac{k_F m}{2\pi^2} \int_0^1 dk k^2 \int_{-1}^1 dz \left\{ \frac{1}{\omega - q^2/2 - kqz} - \frac{1}{\omega + q^2/2 + kqz} \right\}. \tag{19}$$

Performing the z integral we get,

$$G^R_\rho(\omega, q) = \frac{k_F m}{2\pi^2 q} \int_0^1 dk k \left\{ \ln \left| \frac{\omega/q - q/2 + k}{\omega/q - q/2 - k} \right| - \ln \left| \frac{\omega/q + q/2 + k}{\omega/q + q/2 - k} \right| \right\}.$$

Now, using the identity

$$\int_0^1 dx x \ln(a + x) = \frac{1}{2} \left(1 - a^2\right) \ln|a + 1| + \frac{a^2}{2} \ln|a| - \frac{1}{4} + \frac{a}{2},$$

we get the final expression,

$$\begin{aligned} G^R_\rho(\omega, q) &= \frac{k_F m}{2\pi^2} \left[-1 + \frac{1}{2q} \left\{ 1 - (\omega/q - q/2)^2 \right\} \ln \left| \frac{1 + (\omega/q - q/2)}{1 - (\omega/q - q/2)} \right| \right. \\ &- \left. \frac{1}{2q} \left\{ 1 - (\omega/q + q/2)^2 \right\} \ln \left| \frac{1 + (\omega/q + q/2)}{1 - (\omega/q + q/2)} \right| \right], \end{aligned} \tag{20}$$

where all the variables are dimensionless.

Next we evaluate the imaginary part, which for $\omega > 0$ is,

$$G^I_\rho(\omega, q) = -2\pi \sum_k n_k (1 - n_{k+q}) \delta(\omega - \epsilon_{k+q} + \epsilon_k).$$

In the continuum limit, the integral, in terms of dimensionless variables, can be written as,

$$G^I_\rho(\omega, q) = -\frac{k_F m}{2\pi} \int_0^1 dk k^2 \int_{-1}^1 dz \theta(\left|\vec{k} + \vec{q}\right| - 1) \delta(\omega - q^2/2 - kqz). \tag{21}$$

The above integral is tricky because we have to satisfy both the delta-function and the theta-function restrictions. Since, $|kz| < 1$, we get from the delta-function, $-q + q^2/2 \leq \omega \leq q + q^2/2$. This is a necessary condition. Next we note that whenever the delta-function is satisfied, the z-integral gives a factor of $1/kq$, so that we are left with

$$G^I_\rho(\omega, q) = -\frac{k_F m}{2\pi q} \int_{k_{\min}}^{k_{\max}} dk k, \tag{22}$$

where $k_{\max}$ and $k_{\min}$ are determined by the delta-function or the theta- function. It is easy to see that $k_{\max} = 1$. To determine $k_{\min}$ we note that the theta-function gives $(k^2 + q^2 + 2kqz) > 1$, while from the delta-function we have $qkz = \omega - q^2/2$. From these we get, $k_{\min} = (1 - 2\omega)^{1/2}$. This is the $k_{\min}$ from the theta-function. Now in the delta-function, since $|z| \leq 1$, we have $-qk + q^2/2 \leq \omega \leq qk + q^2/2$. This gives $k_{\min} = |\omega/q - q/2|$.

This is $k_{\min}$ from the delta-function. We have to choose the larger of the two $k_{\min}$. It is easy to see that the condition when the second $k_{\min}$ dominates is when $\omega > q - q^2/2$. Thus, finally we have, for $0 < \omega < q - q^2/2$, and $q < 2$ (since $\omega > 0$),

$$G^I_\rho(\omega, q) = -\frac{k_F m}{2\pi q} \int_{(1-2\omega)^{1/2}}^1 dkk = -\frac{k_F m\omega}{2\pi q}, \tag{23}$$

and for $q - q^2/2 < \omega < q + q^2/2$,

$$G^I_\rho(\omega, q) = -\frac{k_F m}{2\pi q} \int_{(|\omega/q - q/2|}^1 dkk = -\frac{k_F m}{2\pi q} \left\{1 - (\omega/q - q/2)^2\right\}. \tag{24}$$

2. $G_\rho(\omega, q)$ **in 2 dimensions**

In dimensionless parameters the real part is,

$$G^R_\rho(\omega, q) = \frac{m}{2\pi^2} \int_0^1 dkk \int_0^{2\pi} d\phi \left\{\frac{1}{\omega - q^2/2 - kq\cos\phi} - \frac{1}{\omega + q^2/2 + kq\cos\phi}\right\}.$$

The ϕ integral can be done using the identity

$$\int_0^{2\pi} \frac{d\phi}{a + b\cos\phi} = \frac{2\pi}{\sqrt{a^2 - b^2}}. \tag{25}$$

Then we have,

$$\begin{aligned} G^R_\rho(\omega, q) &= \frac{m}{\pi} \int_0^1 dkk \left\{\frac{1}{\sqrt{(\omega - q^2/2)^2 - k^2q^2}} - \frac{1}{\sqrt{(\omega + q^2/2)^2 - k^2q^2}}\right\} \\ &= -\frac{m}{\pi q} \left\{\sqrt{(\omega/q - q/2)^2 - 1} - \sqrt{(\omega/q + q/2)^2 - 1}\right. \\ &+ \left. |\omega/q + q/2| - |\omega/q - q/2|\right\} \end{aligned} \tag{26}$$

The imaginary part can be written as,

$$\begin{aligned} G^I_\rho(\omega, q) &= -\frac{m}{2\pi} \int_0^1 dkk \int_0^{2\pi} d\phi\theta(\left|\vec{k} + \vec{q}\right| - 1)\delta(\omega - q^2/2 - kq\cos\phi) \\ &= -\frac{m}{2\pi} \int_0^1 dkk \int_{-1}^1 dz \frac{1}{\sqrt{1 - z^2}} \theta(k^2 + q^2 + 2kqz - 1)\delta(\omega - q^2/2 - kqz). \end{aligned} \tag{27}$$

The arguement now is similar to that in 3 dimensions. The integral over z gives a factor of $1/kq$ and the k integral is restricted to $k_{\min}$ and $k_{\max} = 1$. Also since, $z = (\omega - q^2/2)/kq$, we have,

$$\begin{aligned} G^I_\rho(\omega, q) &= -\frac{m}{2\pi q} \int_{k_{\min}}^1 dkk \frac{1}{\sqrt{k^2 - (\omega/q - q/2)^2}} \\ &= -\frac{m}{2\pi q} \left\{\sqrt{1 - (\omega/q - q/2)^2} - \sqrt{k^2_{\min} - (\omega/q - q/2)^2}\right\}. \end{aligned} \tag{28}$$

Now, for $0 \le \omega \le q - q^2/2$, and $q < 2$, $k_{\min} = (1-2\omega)^{1/2}$. Then,

$$G^I_\rho(\omega, q) = -\frac{m}{2\pi q}\left\{\sqrt{1-(\omega/q - q/2)^2} - \sqrt{1-(\omega/q + q/2)^2}\right\}, \tag{29}$$

and for $q - q^2/2 \le \omega \le q + q^2/2$,

$$G^I_\rho(\omega, q) = -\frac{m}{2\pi q}\sqrt{1-(\omega/q - q/2)^2}. \tag{30}$$

3. $G_\rho(\omega, q)$ **in 1 dimension**

$$\begin{aligned} G^R_\rho(\omega, q) &= \frac{m}{k_F \pi}\int_{-1}^{1} dk \left\{\frac{1}{\omega - q^2/2 - kq} - \frac{1}{\omega + q^2/2 + kq}\right\} \\ &= -\frac{m}{k_F \pi q}\left\{\ln\left|\frac{1+(\omega/q + q/2)}{1-(\omega/q+q/2)}\right| + \ln\left|\frac{1-(\omega/q - q/2)}{1+(\omega/q - q/2)}\right|\right\}, \end{aligned} \tag{31}$$

and

$$\begin{aligned} G^I_\rho(\omega, q) &= -\frac{m}{k_F}\int_{1-q}^{1} dk\,\delta(\omega - q^2/2 - kq) \\ &= -\frac{m}{k_F q} \quad \text{for } q - q^2/2 \le \omega \le q + q^2/2, \\ &= 0 \quad \text{otherwise}. \end{aligned} \tag{32}$$

Evaluation of Matsubara Sums (Problem 3)

The result of the two Matsubara sums $\frac{1}{\beta}\sum_n \frac{e^{i\omega_n 0^+}}{i\omega_n - \epsilon_k}$ and $\frac{1}{\beta}\sum_n \frac{e^{i\omega_n 0^-}}{i\omega_n - \epsilon_k}$ can be easily guessed if we remember the definition of Fourier transform of temperature Green's function. Since,

$$\begin{aligned} g(k, \tau) &= -\langle\, T_\tau c_k(\tau) c_k^\dagger(0)\,\rangle \\ &= \frac{1}{\beta}\sum_n e^{-i\omega_n \tau} g(k, i\omega_n), \end{aligned} \tag{33}$$

and

$$g(k, i\omega_n) = \frac{1}{i\omega_n - \epsilon_k}, \tag{34}$$

we can easily write,

$$\frac{1}{\beta}\sum_n \frac{e^{i\omega_n 0^+}}{i\omega_n - \epsilon_k} = \langle\, c_k^\dagger(0) c_k(0^-)\,\rangle = f_k, \tag{35}$$

and

$$\frac{1}{\beta}\sum_n \frac{e^{i\omega_n 0^-}}{i\omega_n - \epsilon_k} = -\langle\, c_k(0^+) c_k^\dagger(0)\,\rangle = -(1 - f_k) \tag{36}$$

From this discussion it is obvious that depending on whether $\tau > 0$ or $\tau < 0$, the Green's function represents particle excitation or hole excitation respectively. This is the reason why the two sums differ.

However, now we will evaluate the sums by the usual method as an exercise in doing Matsubara sums. In particular, we will evaluate the expression,

$$g((k,\tau) = \frac{1}{\beta}\sum_n \frac{e^{-i\omega_n\tau}}{i\omega_n - \epsilon_k},$$

for $\tau > 0$ and $\tau < 0$. The important observation is that the Fermi function $f(z) = 1/(e^{\beta z}+1)$, has poles at fermionic Matsubara frequencies $\omega_n = (2n+1)\pi/\beta$, with residue $-1/\beta$. Then, the integral (refer to figure 1)

$$-\oint_C \frac{dz}{2\pi i}\left(\frac{1}{e^{\beta z}+1}\right)\left(\frac{1}{z-\epsilon_k}\right)e^{-z\tau} = \frac{1}{\beta}\sum_n \frac{e^{-i\omega_n\tau}}{i\omega_n - \epsilon_k}. \tag{37}$$

The integrand in the above expression has another pole at $z = \epsilon_k$. The contour C can now be deformed into the contour C' (refer to figure 2). Thus we can write,

$$\frac{1}{\beta}\sum_n \frac{e^{-i\omega_n\tau}}{i\omega_n - \epsilon_k} = -\oint_{C'} \frac{dz}{2\pi i}\left(\frac{1}{e^{\beta z}+1}\right)\left(\frac{1}{z-\epsilon_k}\right)e^{-z\tau}. \tag{38}$$

Now, if $\tau < 0$, as $|z| \to \infty$, the integrand $\sim \frac{1}{z}e^{-(\beta+\tau)z}$, for $Re(z) > 0$. This is exponentially supressed because $\tau > -\beta$, always. And for $Re(z) < 0$, the integrand $\sim \frac{1}{z}e^{-z\tau}$, which is also exponentially supressed. Thus, in the contour C' the contribution from the large arcs is zero, and the integral in the above expression only picks up the pole at $z = \epsilon_k$. Thus, for $\tau < 0$,

$$\frac{1}{\beta}\sum_n \frac{e^{-i\omega_n\tau}}{i\omega_n - \epsilon_k} = e^{-\epsilon_k\tau}f_k, \tag{39}$$

which confirms the result of equation (35). Now for $\tau > 0$, we cannot use the same arguement as above because for $Re(z) < 0$, the integrand $\sim \frac{1}{z}e^{-z\tau}$ will diverge exponentially. So, we now introduce the function $e^{\beta z}/(e^{\beta z}+1)$, which also has poles at fermionic Matsubara frequencies, with residue $+1/\beta$, and which is able to supress the above mentioned divergence. Then we can write, for $\tau > 0$,

$$\begin{aligned}\frac{1}{\beta}\sum_n \frac{e^{-i\omega_n\tau}}{i\omega_n - \epsilon_k} &= \oint_C \frac{dz}{2\pi i}\left(\frac{e^{\beta z}}{e^{\beta z}+1}\right)\left(\frac{1}{z-\epsilon_k}\right)e^{-z\tau} \\ &= \oint_{C'} \frac{dz}{2\pi i}\left(\frac{e^{\beta z}}{e^{\beta z}+1}\right)\left(\frac{1}{z-\epsilon_k}\right)e^{-z\tau}, \\ &= -(1-f_k)e^{-\epsilon_k\tau}\end{aligned} \tag{40}$$

where in the last step we have used the fact that with this choice of function the integrand vanishes over the large arcs of the contour C', and so picks up only the pole at $z = \epsilon_k$. This confirms the result in equation (36).

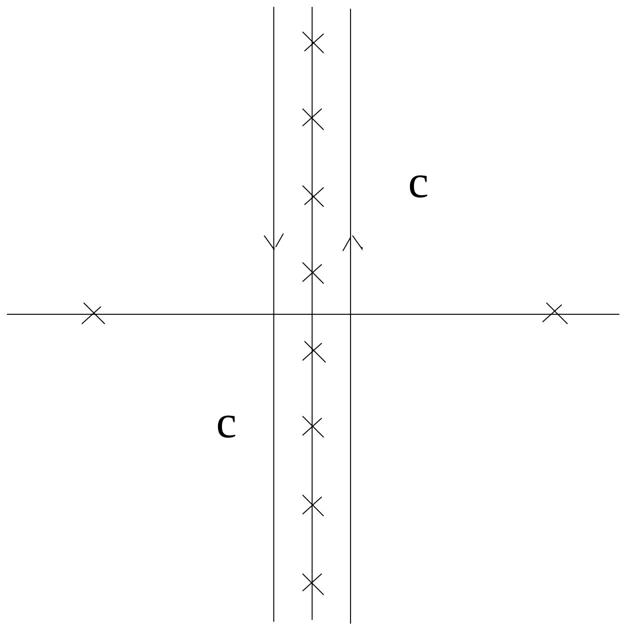

Figure 1: Initial contour C. It picks up the poles at fermionic Matsubara frequencies.

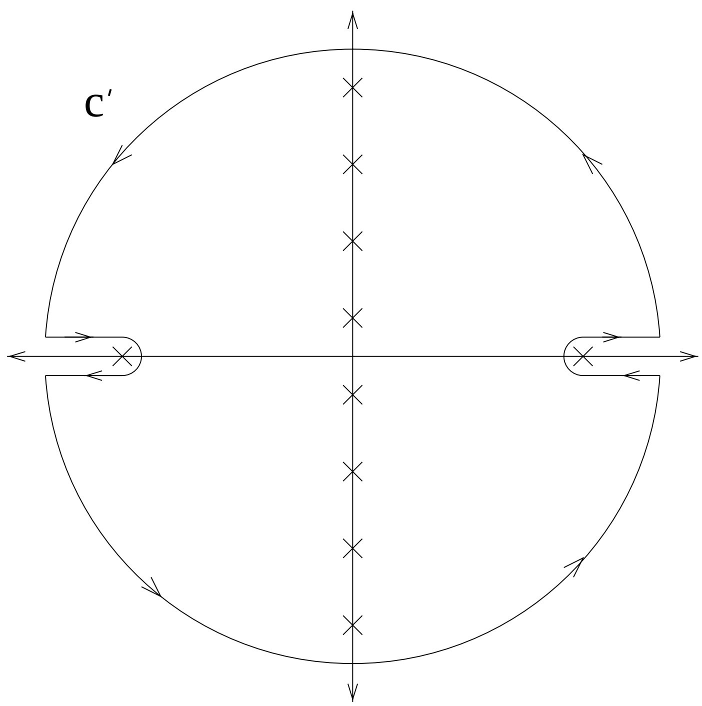

Figure 2: Deformed contour. The contribution from the arcs is zero.