Problem Set 5, If you have handed in 4 homeworks, you need not bother with this

Problem set 5: a new look way of looking at problem set 4

Now that you know about Wick's theorem, you have another way way of redoing the first part of the previous homework by a direct application of that idea. Rederive your expressions using the following hints:

Define

$$D_\rho(t,q) = -i\langle T(\rho_q(t)\rho_q^\dagger(0))\rangle$$

$$D_\Delta(t,q) = -i\langle T(\Delta_q(t)\Delta_q^\dagger(0))\rangle$$

$\rho_q(t) = \sum_{k\sigma} c^{\dagger\sigma}_{k+q} c_{k\sigma}$; $\quad {\Delta^\dagger}_q(t) = \sum_k c_{k+q\downarrow} c_{k\uparrow}$; $\quad \langle ... \rangle$ is with respect to a Fermi sea.

a) Use Wick's theorem to express $D_\rho(t,q)$ and $D_\Delta(t,q)$ in terms of $-i\langle T(c_k(t)c_k^\dagger(0))\rangle = G(k,t)$

b) Compute

$$G(k,\omega) = \int_{-\infty}^{\infty} e^{i\omega t} G(k,t)dt$$

- show it is given by

$$G(k,\omega) = \frac{1}{i\omega - \varepsilon_k + i\delta sign(\mid k \mid - k_F)}, \quad \delta > 0, \quad \delta \to 0.$$

c) Combine (a) + (b) and use the fact that the Fourier transform of the product is the solution of the Fourier transform to show that

$$D_\rho(\omega,q) = \int_{-\infty}^{\infty} dt e^{i\omega t} D_\rho(t,q)$$

$$D_\Delta(\omega,q) = \int_{-\infty}^{\infty} dt e^{i\omega t} D_\Delta(t,q)$$

are given by expressions like

$$D_\rho(\omega,q) \sim -i \int d\omega' \sum_k G(\omega' + \omega, k+q) G(\omega', k)$$

$$D_\Delta(\omega,q) \sim -i \int d\omega' \sum_k G(-\omega' + \omega, k+q) G(\omega', k)$$

d) Now carry out the frequency integral using the residue theorem. Get expressions of the functions

$$D_\rho(\omega, q) \sim \sum_k \frac{[f(\varepsilon_{k+q}) - f(\varepsilon_k)]}{\varepsilon_{k+q} - \varepsilon_k - \omega - i\delta}, \quad \omega > 0$$

$$D_\Delta(\omega, q) \sim \sum_k \frac{[1 - (f(\varepsilon_{k+q}) + f(\varepsilon_k))]}{\varepsilon_{k+q} + \varepsilon_k - \omega - i\delta}, \quad \omega > 0$$

$$f(\varepsilon_k) = \begin{cases} 1 & \text{if } k < k_F \\ 0 & \text{if } k > k_F \end{cases}$$

e) Now everything is reduced to a d dimensional integral over k. One way to do them is say in the case of D_ρ

$$D_\rho \sim \sum_k f(\varepsilon_k) \left[\frac{1}{\varepsilon_k - \varepsilon_{k+q} - \omega - i\delta} - \frac{1}{\varepsilon_{k+q} - \varepsilon_k - \omega - i\sigma} \right]$$

Introduce polar coordinates and integrate over $d\theta$ and over dk, $0 \leq k \leq k_F$. The 3d integral is done in Fetter and Walecka page 158-162.

If you want, you can calculate the D_ρ and $\mathrm{Im} D_\rho$ separately.

In $1d$ the integrals are even easier; $2d$ is more fun.

Repeat the same for D_Δ.

Good Luck