1 Functional Integrals for Fermions

This approach can be extended to Fermions in two formally similar but conceptually inequivalent ways :–

In the first we define coherent states of Fermions by allowing to take linear combination of states with coefficients valued in a Grassmann algebra, i.e., the algebra generated by $2N$ anticommuting objects $\{\xi_i \xi_i^*; i = 1, \cdots N\}$. Any member of this algebra is called a Grassmann number. $\int d\xi_i \equiv \frac{\partial}{\partial \xi_i}$ anticommutes with ξ_j, $j \neq i$

We define a fermion coherent state as

$$|\xi\rangle \equiv \exp\left(-\sum_\alpha \xi_\alpha a_\alpha^\dagger |0\rangle\right) = \prod_\alpha \left(1 - \xi_\alpha a_\alpha^\dagger\right)|0\rangle$$

We require that $\{\xi, a\} = 0$ and $(\xi a)^\dagger = a^\dagger \xi^*$

Then it is clear that

$$a_\alpha |\xi\rangle = \xi_\alpha |\xi\rangle$$
$$\langle \xi | a_\alpha^\dagger = \langle \xi | \xi_\alpha^*$$

To understand the properties of the Fermionic coherent states consider first a single Fermion degree of freedom $\{a, a^\dagger\}$. Define

$$|\xi\rangle \equiv e^{-\xi a^\dagger} |0\rangle$$

$\{\xi, a\} = 0$ and $(\xi a)^\dagger = a^\dagger \xi^*$. The overcompleteness relations are easily worked out

$$\begin{aligned}
\int d\xi^* d\xi \, |\xi\rangle \langle \xi| &= \frac{\partial}{\partial \xi^*}\frac{\partial}{\partial \xi}(|0\rangle - \xi a^\dagger |0\rangle)(\langle 0| - \langle 0| a \xi^*)(1 - \xi^* \xi) \\
&= |0\rangle \langle 0| + a^\dagger |0\rangle \langle 0| a = I
\end{aligned}$$

Notice that ξ anticommutes with vectors in the Hilbert space containing an odd number of Fermions in which case

$$\xi |\psi\rangle = -|\psi\rangle \xi$$

$$\begin{aligned}
\langle \eta | \xi \rangle &= (\langle 0| - \langle 0| a \eta^*)(|0\rangle - \xi a^\dagger |0\rangle) \\
&= 1 + \eta^* \xi = e^{\eta^* \xi}
\end{aligned}$$

Finally, notice that if $|\psi_i\rangle$ are vectors in the Fock space with a definite number of Fermions then $\langle \xi | \psi_i \rangle$ is a grassmann variable which contains an odd number of η_α if $|\psi_i\rangle$ has an odd number of Fermions and contains an even number of η_α if $|\psi_i\rangle$ has an even number of Fermions. Hence:

$$\begin{aligned}
\langle \xi | \psi_i \rangle \langle \psi_j | \xi \rangle &= \langle \psi_j | \xi \rangle \langle \xi | \psi_i \rangle (-1)^F \\
&= \langle \psi_j | \xi \rangle \langle \psi_i | -\xi \rangle = \langle \psi_j | -\xi \rangle \langle \psi_i | \xi \rangle
\end{aligned}$$

This gives an expression for the trace of an operator A which acts on the Fock space and preserves the number of particles.

$$\begin{aligned}
\mathrm{TrA} &= \sum_n \langle n|A|n\rangle = \sum_n \int \prod_\alpha d\xi_\alpha^* d\xi_\alpha e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} \langle n|\xi\rangle \langle \xi|A|n\rangle \\
&= \int \prod_\alpha d\xi_\alpha^* d\xi_\alpha e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} \sum_n \langle -\xi|A|n\rangle \langle n|\xi\rangle \\
&= \int \prod_\alpha d\xi_\alpha^* d\xi_\alpha e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} \langle -\xi|A|\xi\rangle
\end{aligned}$$

Equipped with these techniques we can write functional integral expression for Fermi systems in complete analogy with the Bosonic theory.

$$U(\tau,\tau') = \prod_{i=0}^{N-1} U(\tau_{i+1},\tau_i)$$

$\tau_0 = \tau^{'}, \tau_N = \tau$. We express

$$U(\tau_{i+1},\tau_i) = \exp -H\left(\frac{\tau_{i+1}+\tau_i}{2}\right)\Delta\tau$$

and insert at each point a resolution of the identity to find

$$\langle \eta_N \left| U(\tau,\tau') \right| \eta_0\rangle = \prod_{i=1}^{N-1} \int d\eta_i^* d\eta_i e^{-\eta_i^*\eta_i} \langle \eta_{i+1} \left| U(\tau_{i+1},\tau_i) \right| \eta_i\rangle \langle \eta_1 \left| U(\tau_1,\tau_0) \right| \eta_0\rangle$$

Using the expression for the trace of an operator derived earlier one finds

$$\mathrm{TrU}(\tau,\tau') = \int \mathrm{d}\eta_0^* \mathrm{d}\eta_0 \mathrm{e}^{-\eta_0^*\eta_0} \langle \eta_0 \left| \mathrm{U}(\tau,\tau') \right| \eta_0\rangle$$

If $U \simeq e^{-H\left(\frac{\tau_{i+1}+\tau_i}{2}\right)\Delta\tau}$ and $H_c(\eta^*,\eta)$ with η^* and η and replaced by $a^\dagger$ and a give the normal ordered form of H we find

$$\begin{aligned}
\mathrm{TrU}(\tau,\tau') &= \prod_{i=0}^{N-1} \int d\eta_i^* d\eta_i e^{-\sum_{i=0}^{N-1} \eta_i^*\eta_i} \langle \eta_{i+1}|\eta_i\rangle e^{-H_c\left(\eta_{i+1}^*,\eta_i\right)\Delta\tau} \\
&= \prod_{i=0}^{N-1} \int d\eta_i^* d\eta_i e^{-\sum_{i=0}^{N-1} \eta_i^*\eta_i} e^{\sum_{i=0}^{N-1} \eta_i^*\eta_i} e^{-H_c \sum_{i=0}^{N-1}\left(\eta_{i+1}^*,\eta_i\right)\Delta\tau}
\end{aligned}$$

$(\eta_N^* \equiv -\eta_0^*)$. Notice that if H_c contained some explicit time dependence due to the source or an auxillary field, it should be evaluated at time $\tau = \frac{\tau_{i+1}+\tau_i}{2}$. Proceeding heuristically we could define

$$\mathrm{Tr}e^{-\beta \mathrm{H}} = \int \mathrm{D}\eta^* \mathrm{D}\eta \mathrm{e}^{-\left(\eta^* \frac{\partial\eta}{\partial\tau} + \mathrm{H_c}(\eta^*,\eta)\right)}$$

where the operator $\frac{\partial}{\partial\tau}$ is defined on the space of functions obeying $\eta(\beta) \equiv -\eta(0)$.

2 The Standard Gaussian Integral:

$$\int \prod_i d\eta_i^* d\eta_i e^{-\eta^* A\eta+\eta^* j+j^*\eta} = \int \prod_i d\eta_i^* d\eta_i e^{-\eta^* A\eta} e^{j^* A^{-1} j}$$

using the transformation

$$\begin{aligned}\eta &\rightarrow \eta + A^{-1} j \\ \eta^* &\rightarrow \eta^* + j^* A^{-1}\end{aligned}$$

Thus

$$\int \prod_i d\eta_i^* d\eta_i e^{-\eta^* A\eta+\eta^* j+j^*\eta} \simeq (\det A) e^{j^* A^{-1} j}$$

expanding to second order.

$$\begin{aligned}\eta_i^* j_i j_k^* \eta_k &= j_k^* A_{kl}^{-1} j_l \\ \langle \eta_k | \eta_l^* \rangle &= A_{kl}^{-1}\end{aligned}$$