Functional Integrals

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The central problem in quantum mechanics is the calculation of matrix elements of the evolution operator $U(t,t')$, the evaluation of the partition function and other physical observables. Feynman wrote a closed expression for these quantities as a sum over all paths in a suitable function space. This approach is useful because it suggests new insights into a problem, it allows for different types of approximation techniques which are not transparent in the operator formalism and it lends itself (sometimes) to numerical evaluation using Monte Carlo methods.

There are three basic steps in the Feynman construction.

1. First using the group property along any path one can divide the time evolution into a large sequence of small steps.

$$U(t,t') = U(t_N,t_0) = U(t_N,t_{N-1})U(t_{N-1},t_{N-2})\cdot\cdots\cdot U(t_1,t_0) \tag{1}$$

with $t_N = t'$, $t_0 = t$ and $t_i - t_{i-1} = \Delta t \to 0$, N$\to\infty$.

A similar expression applies to the partition function or to the evaluation of correlation functions of an arbitrary observable at times $\tau_1\cdots\cdot\tau_N$. For example

$$Z = \mathrm{Tr}\exp(-\beta \mathrm{H}) = \mathrm{Tr}(\mathrm{e}^{-\Delta\tau\mathrm{H}}\cdots\cdot\mathrm{e}^{-\Delta\tau\mathrm{H}}) \tag{2}$$

with $\Delta\tau = \beta/N$.

2. Insertion of a complete set of states or resolution of the identity.

$$I = \int d\mu(z,z^*)\,|z\rangle\,\langle z| \tag{3}$$

The nature of the functional integral depends on the choice of complete or overcomplete basis obeying Eq.3.

$$\begin{aligned}
\mathrm{Tr}\exp(-\beta\mathrm{H}) &= \mathrm{Tr}\prod_{\mathrm{i}=\{\mathrm{N}-1,\ldots1\}}\mathrm{e}^{-\Delta\tau\mathrm{H}}\int \mathrm{d}\mu(\mathrm{z_i},\mathrm{z_i^*})\,|\mathrm{z_i}\rangle\,\langle\mathrm{z_i}| \\
&= \prod_{i=1}^{N-1}\int d\mu(z_i,z_i^*)\frac{\langle z_{i+1}\,|e^{-\Delta\tau H}|\,z_i\rangle}{\langle z_{i+1}|z_i\rangle}\,\langle z_{i+1}|z_i\rangle
\end{aligned}$$

where we have set $z_N \equiv z_1$.

3. Finally we need expressions for the matrix elements of $e^{-\Delta\tau H}$ in the short time approximation

$$\frac{\left\langle z' \left| e^{-\Delta\tau \hat{H}} \right| z \right\rangle}{\langle z'|z\rangle} \simeq 1 - \frac{\left\langle z' \left| \hat{H} \right| z \right\rangle}{\langle z'|z\rangle} \Delta\tau \simeq e^{-H[z,z']\Delta\tau} \tag{4}$$

where $H[z,z'] = \frac{\langle z'|\hat{H}|z\rangle}{\langle z'|z\rangle}$ is often denoted the symbol of the operator $\hat{H}$.

The overall result is

$$Z = \prod_{i=1}^{N-1} \int d\mu(z_i, z_i^*) e^{-\Delta\tau H[z_{i+1},z_i]} \langle z_{i+1}|z_i\rangle \tag{5}$$

which for one harmonic oscillator becomes

$$\begin{aligned} Z &= \prod_{i=1}^{N-1} e^{\sum_i (z_{i+1}^* - z_i^*) z_i - \Delta\tau H[z_{i+1}^*, z_i]} \\ &\simeq \int \mathcal{D}z^* \mathcal{D}z e^{\int_0^\beta \frac{\partial}{\partial\tau} z^*(\tau) z(\tau) - H[z^*(\tau), z(\tau)] d\tau} \end{aligned}$$