Problem set 1 (version Wed Sep 20) (Due on Thursday September 28th)

1. Consider the simple harmonic oscillator

$$\hat{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Write it in "second quantized" form , by expressing x and p in terms of creation and annihilation operators.

$$\hat{H} = \hbar\omega(a^\dagger a + \frac{1}{2}).$$

(i) Derive the equation of motion for a and show that

$$a(t) = a(0)e^{-i\omega t}.$$

Repeat this for $a^\dagger$.
(ii) Using the relationship

$$a^\dagger = \frac{1}{\sqrt{2\hbar}}\left[\frac{\hat{p}}{\sqrt{m\omega}} + i\hat{x}\sqrt{m\omega}\right].$$

use the results of (i) to show that

$$\begin{aligned}\hat{p}(t) &= \hat{p}(0)\cos\ \omega t - m\omega\hat{x}(o) sin\ \omega t\\ \hat{x}(t) &= \hat{x}(0)\cos\ \omega t + \frac{\hat{p}(0)}{m\omega} sin\ \omega t.\end{aligned}$$

(iii) The above operator expressions appear identical to the classical equations of motion.

Why?

2. Show that the Heisenberg spin operator S can be written in second quantized form as

$$\begin{aligned}S_x &= \frac{1}{2}(c^\dagger_\uparrow c_\downarrow + c^\dagger_\downarrow c_\uparrow),\\ S_y &= -\frac{i}{2}(c^\dagger_\uparrow c_\downarrow - c^\dagger_\downarrow c_\uparrow),\\ S_z &= \frac{1}{2}(c^\dagger_\uparrow c_\uparrow - c^\dagger_\downarrow c_\downarrow).\end{aligned}$$

Check the commutation relations!

3. Consider a system of fermions created by the field $\psi^{\dagger}(\vec{r})$ interacting under the Yukawa potential

$$V(r) = \frac{Ae^{-\lambda r}}{4\pi r}.$$

(i) Write the Hamiltonian in second quantized form, using the position basis.

(ii) Write the Hamiltonian in second quantized notation in the momentum basis, where

$$c_{\vec{k}}{}^{\dagger} = \int d^3 r \psi^{\dagger}(\vec{r}) e^{i\vec{k}.\vec{r}}.$$

You will find it helpful to derive the Fourier representation

$$V(r) = \int \frac{d^3 q}{(2\pi)^3} e^{i\vec{q}.\vec{r}} \frac{A}{(q^2+\lambda^2)}.$$

4. For the spin 1/2 fermions considered in question 3, prove the "Hubbard Stratonovich" identity

$$e^{-a(n_{\uparrow}-1/2)(n_{\downarrow}-1/2)} = \frac{1}{2} e^{-a/4} \sum_{\sigma=\pm 1} e^{a\sigma(n_{\uparrow}-n_{\downarrow})}$$

where $\cosh a = e^{\frac{a}{4}}$ (Hint: consider the matrix representation of the operator on the left hand side in the four dimensional Fock space of the up and down electrons.) We shall later see how, considering $a \sim \delta t$ to be an infinitesimal time interval, we can use this relation to treat interacting spins as spins moving in a fluctuating Ising magnetic field.

6. Let $a^{\dagger}_{\alpha}$, a_{α} be Fermion creation and annihilation operators, and assume

$$H_o = \sum_{\alpha} \epsilon_{\alpha} a^{\dagger}_{\alpha} a_{\alpha}.$$

(i) Compute $Tr[e^{\beta H_o} a^{\dagger}_{\alpha} a_{\alpha}]$

(ii) Compute Tr $[e^{-\beta H_o} a_{\alpha} a^{\dagger}_{\alpha}]$.

(iii) Compute $Tr[e^{-\beta H_o} a_\alpha a_\alpha^\dagger]$. Using the identity $e^{-\beta H_o} a_\alpha^\dagger a_\alpha = -T \frac{\partial}{\partial \epsilon_\alpha} e^{-\beta H_o}$ check your answer to (i).

(iv) Confirm that the expectation

$$\langle n_a \rangle = Z^{-1} Tr[e^{-\beta H_o} a_\alpha^\dagger a_\alpha] = f(\epsilon_\alpha)$$

where $f(x)$ is the Fermi function.

(v) Repeat the above procedure assuming that $a_a^\dagger$ and a_a are boson operators. What are the restrictions on the values of ϵ_a and what happens when one of these energies ϵ_a becomes zero?

(vi) Calculate

$$\begin{pmatrix} a_\alpha^\dagger(\tau) \\ a_\alpha(\tau) \end{pmatrix} = e^{\tau H_o} \begin{pmatrix} a_\alpha^\dagger \\ a_\alpha \end{pmatrix} e^{-\tau H_o}$$