These notes are to document the nitty gritty of the emittance exchange matrixes using thin lens 4x4 matrixes x, x', z, z'.

Basic thin lens elements

The elements are defined below. The bend element is a thin lens where q is zero under the condition of a pole tip with a box or trapezoidal (parallel sides) shape. (i.e. a sector magnet and other edge angles give non zero q bend)

Quad $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ q & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Drift $$\begin{pmatrix} 1 & l & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Bend with focusing $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ q_{bend} & 1 & 0 & \theta \\ -\theta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Cavity $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & T & 0 \\ 0 & 0 & 1 & 0 \\ T & 0 & 0 & 1 \end{pmatrix}$$

We assume that we want to place the cavity in a region with dispersion but without dispersion prime. Further we will set the deflecting mode cavity strenngth so that it supplies acceleration/decelleration voltage as a function of x in the dispersion section so as to cancel the dp/dx generated by the dispersion.

This all follows the Kim, Emma work and half of a chichane provides these conditions.
In what follows we will derive the matrix up to the cavity, propogate through the cavity, and then ask what matrix is necessary after the cavity to accomplish the x<>z exchange and result in a transport matrix of the form

$$\begin{pmatrix} 0 & 0 & ? & ? \\ 0 & 0 & ? & ? \\ ? & ? & 0 & 0 \\ ? & ? & 0 & 0 \end{pmatrix}$$

The 1/2 chicane dogleg

Dogleg

Drift, Bend1, Drift2, Bend2, Drift3>

$$\begin{pmatrix} 1 & l_3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ q_{b2} & 1 & 0 & \theta_2 \\ -\theta_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & l_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ q_{b1} & 1 & 0 & \theta_1 \\ -\theta_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & l & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1+q_1(l_2+l_3)+q_2l_3+q_1q_2l_2l_3 & l_1+l_2+l_3+q_1l_1(l_2+l_3)+q_2l_3(l_1+l_2)+q_1q_2l_1l_2 & 0 & \theta_1(l_2+l_3)+\theta_2l_3+\theta_1q_2l_2l_3 \\ q_1+q_2+q_1q_2l_2 & 1+q_1l_1+q_2(l_1+l_2)+q_1q_2l_1l_2 & 0 & \theta_1+\theta_2+\theta_1q_2l_2 \\ -\theta_1-\theta_2-\theta_2l_2q_1 & -\theta_1l_1-\theta_2(l_1+l_2)-\theta_2l_1l_2q_1 & 1 & -\theta_1\theta_2l_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Choose $\theta_2 = -\theta_1$ and make disp prime $\partial x'/\partial z' = 0$, this says q2=0

Dogleg matrix becomes

$$\begin{pmatrix} 1+q_1(l_2+l_3) & l_1+l_2+l_3+q_1l_1(l_2+l_3) & 0 & \theta_1l_2 \\ q_1 & 1+q_1l_1 & 0 & 0 \\ \theta_1l_2q_1 & \theta_1l_2(1+l_1q_1) & 1 & \theta_1^{\,2}l_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We label this matrix

$$\begin{pmatrix} A & B & 0 & \partial x/\partial z' \\ C & D & 0 & 0 \\ E & F & 1 & \partial z/\partial z' \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A & B & 0 & \partial x/\partial z' \\ C & D & 0 & 0 \\ C\partial x/\partial z' & D\partial x/\partial z' & 1 & \theta_1\partial x/\partial z' \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\partial x/\partial z'$ is dispersion and $\partial z/\partial z'$ (R56) is the z compression.

Multiply the dogleg by the cavity matrix.. Dogleg, cavity.>

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & T & 0 \\ 0 & 0 & 1 & 0 \\ T & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A & B & 0 & \partial x/\partial z' \\ C & D & 0 & 0 \\ C\partial x/\partial z' & D\partial x/\partial z' & 1 & \theta_1\partial x/\partial z' \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} A & B & 0 & \partial x/\partial z' \\ C+TC\partial x/\partial z' & D+TD\partial x/\partial z' & T & T\partial z/\partial z' \\ C\partial x/\partial z' & D\partial x/\partial z' & 1 & \partial z/\partial z' \\ TA & TB & 0 & 1+T\partial x/\partial z' \end{pmatrix}$$

Now try to find the matrix that will provide the x<>z interchange.
Assume it has the same general properties as the dogleg, and multiply the preceeding matrix by it.

$$\begin{pmatrix} \alpha & \beta & 0 & \pi \\ \gamma & \delta & 0 & \rho \\ \varepsilon & \zeta & 1 & \sigma \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A & B & 0 & \partial x/\partial z' \\ C+TC\partial x/\partial z' & D+TD\partial x/\partial z' & T & T\partial z/\partial z' \\ C\partial x/\partial z' & D\partial x/\partial z' & 1 & \partial z/\partial z' \\ TA & TB & 0 & 1+T\partial x/\partial z' \end{pmatrix}$$

We want to make the cavity gradient so that $\begin{array}{l} 1+T\partial x/\partial z'=0, \\ T=-1/\partial x/\partial z' \end{array}$

The above dogleg times cavity matrix becomes

$$\begin{pmatrix} A & B & 0 & \partial x/\partial z' \\ C+TC\partial x/\partial z' & D+TD\partial x/\partial z' & T & T\partial z/\partial z' \\ C\partial x/\partial z' & D\partial x/\partial z' & 1 & \partial z/\partial z' \\ TA & TB & 0 & 1+T\partial x/\partial z' \end{pmatrix} = \begin{pmatrix} A & B & 0 & \partial x/\partial z' \\ 0 & 0 & T & T\partial z/\partial z' \\ C\partial x/\partial z' & D\partial x/\partial z' & 1 & \partial z/\partial z' \\ TA & TB & 0 & 0 \end{pmatrix}$$

It is convenient that the (2,1) and (2,2) elements become zero as well.

Now multiply dogleg1, cavity, dogleg2 >

$$\begin{pmatrix} \alpha & \beta & 0 & \pi \\ \gamma & \delta & 0 & \rho \\ \varepsilon & \zeta & 1 & \sigma \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A & B & 0 & \partial x/\partial z' \\ 0 & 0 & T & T\partial z/\partial z' \\ C\partial x/\partial z' & D\partial x/\partial z' & 1 & \partial z/\partial z' \\ TA & TB & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} \alpha A+\pi TA & \alpha B+\pi TB & \beta T & \alpha\partial x/\partial z'+\beta T\partial z/\partial z' \\ \gamma A+\rho TA & \gamma B+\rho TB & \delta T & \gamma\partial x/\partial z'+\delta T\partial z/\partial z' \\ \varepsilon A+C\partial x/\partial z' & \varepsilon B+D\partial x/\partial z'+\sigma TB & \zeta T+1 & \varepsilon\partial x/\partial z'+\partial z/\partial z'(1+\zeta T) \\ TA & TB & 0 & 0 \end{pmatrix}$$

We want to ask that this be

$$\begin{pmatrix} 0 & 0 & ? & ? \\ 0 & 0 & ? & ? \\ ? & ? & 0 & 0 \\ ? & ? & 0 & 0 \end{pmatrix}$$

therefore

$$\alpha/\pi = -T$$
$$\gamma/\rho = -T$$
$$\zeta T = -1$$
$$\varepsilon\partial x/\partial z' + \partial z/\partial z'(1+\zeta T) = 0$$
$$so$$
$$\varepsilon = 0$$

If the matrix after the cavity is another dogleg, the it becomes
The 2nd dogleg matrix

$$\begin{pmatrix} \alpha & \beta & 0 & \pi \\ \gamma & \delta & 0 & \rho \\ \varepsilon & \zeta & 1 & \sigma \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\pi T & \beta & 0 & \pi \\ -\rho T & \delta & 0 & \rho \\ 0 & -1/T & 1 & \sigma \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1+q_1(l_2+l_3)+q_2l_3+q_1q_2l_2l_3 & l_1+l_2+l_3+q_1l_1(l_2+l_3)+q_2l_3(l_1+l_2)+q_1q_2l_1l_2 & 0 & \theta_1(l_2+l_3)+\theta_2l_3+\theta_1q_2l_2l_3 \\ q_1+q_2+q_1q_2l_2 & 1+q_1l_1+q_2(l_1+l_2)+q_1q_2l_1l_2 & 0 & \theta_1+\theta_2+\theta_1q_2l_2 \\ -\theta_1-\theta_2-\theta_2l_2q_1 & -\theta_1l_1-\theta_2(l_1+l_2)-\theta_2l_1l_2q_1 & 1 & -\theta_1\theta_2l_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where this matrix is just copied from above.

$$\theta_2 = -\theta_1$$
$$\varepsilon = 0$$
Again if
$$q_1 = 0$$

That is, the nearest bend to the cavity on either side has q=0 (box or trapezoidal pole tip)

The matrix becomes

$$\begin{pmatrix} -\pi T & \beta & 0 & \pi \\ -\rho T & \delta & 0 & \rho \\ 0 & -1/T & 1 & \sigma \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+q_2l_3 & l_1+l_2+l_3+q_2l_3(l_1+l_2) & 0 & \theta_1l_2(1+l_3q_2) \\ q_2 & 1+q_2(l_1+l_2) & 0 & \theta_1q_2l_2 \\ 0 & \theta_1l_2 & 1 & \theta_1^{\,2}l_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$T = -1/\partial x/\partial z' = -1/\theta_1 l_2$ from the 1st dogleg, and here again $T = -1/\theta_1 l_2$
The sign of the first bends in both doglegs is the same and both second bends have the opposite sign to the 1st bends.

The matrix of this 2nd dogleg becomes

$$\begin{pmatrix} \alpha & \beta & 0 & \alpha(-1/T) \\ \gamma & \delta & 0 & \gamma(-1/T) \\ 0 & -1/T & 1 & \theta_1(-1/T) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$-1/T = \theta_1 l_2 = \partial z/\partial x'$$

where as it was $T = -1/\partial x/\partial z'$ in the first dogleg.

The rules are:
q_B =0 of the inner bend of each dogleg.
$\theta_1 = -\theta_2$ in both doglegs and θ_1 of each is same sign.
$T = -1/\theta_1 l_2$ in both doglegs, so the product $\theta_1 l_2$ is fixed but not the individual θ or l. That is the bends of the 2nd dogleg can be different from those in the first if the l_2s are chosen appropiately.

Now consider a quad, bend matrix.
Drift l, Quad, Drift l_2, Bend, Drift l_3 >

$$\begin{pmatrix}1 & l_3 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ q_{bend} & 1 & 0 & \theta\\ -\theta & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & l_2 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ q & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & l & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} =$$

$$\begin{pmatrix}1+(l_2+l_3)q+l_3q_B(1+l_2q) & l+l_2+l_3+ql(l_2+l_3)+q_Bl_3(l+l_2+qll_2) & 0 & l_3\theta\\ q+q_B(1+l_2q) & 1+ql+q_B(l+l_2+qll_2) & 0 & \theta\\ -\theta(1+ql_2) & -\theta(l+l_2+qll_2) & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

And the reverse Drift l_3, Bend, Drift l_2, Quad, Drift l >

$$\begin{pmatrix}1 & l & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ q & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & l_2 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ q_{bend} & 1 & 0 & \theta\\ -\theta & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & l_3 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} =$$

$$\begin{pmatrix}1+lq+q_B(l+l_2+qll_2) & l+l_2+l_3+ql(l_2+l_3)+q_Bl_3(l+l_2+qll_2) & 0 & \theta(l+l_2+qll_2)\\ q+q_B(1+ql_2) & 1+q(l_2+l_3)+q_B(l_3+ql_2l_3) & 0 & \theta(1+ql_2)\\ -\theta & -\theta l_3 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

Look at the quad then bend matrix first.

$$\begin{pmatrix}1+(l_2+l_3)q+l_3q_B(1+l_2q) & l+l_2+l_3+ql(l_2+l_3)+q_Bl_3(l+l_2+qll_2) & 0 & l_3\theta\\ q+q_B(1+l_2q) & 1+ql+q_B(l+l_2+qll_2) & 0 & \theta\\ -\theta(1+ql_2) & -\theta(l+l_2+qll_2) & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

We want this to be like the matrix that is required after the cavity

$$\begin{pmatrix}\alpha & \beta & 0 & \pi\\ \gamma & \delta & 0 & \rho\\ \varepsilon & \zeta & 1 & \sigma\\ 0 & 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}-\pi T & \beta & 0 & \pi\\ -\rho T & \delta & 0 & \rho\\ 0 & -1/T & 1 & \sigma\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

$\varepsilon = 0 = -\theta(1+ql_2)$

if

$1+ql_2 = 0$

With this the matrix becomes

$$\begin{pmatrix} l_3 q & l_2+l_3+l_3(ql+q_B l_2) & 0 & \theta l_3 \\ q & 1+ql+q_B l_2 & 0 & \theta \\ 0 & -\theta l_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$T = 1/\theta l_2$

$\alpha = -\pi T$

$\gamma = -\rho T$

because

$1+ql_2 = 0$

This matrix can be used as the 2nd half of the transform after a dogleg and cavity.

$[Dogleg] \triangleright [T] \triangleright [Q \triangleright Bend]$

Note that if we define the above $\theta l_2 \equiv \theta' l_2'$ to denote the quad bend system geometry, then $\theta' l_2' = -\theta_1 l_2$ of the 1st bend/2nd drift of the 1st dogleg in the dogleg geometry. And $q' l_2' = -1$

The **rules** are:

q_B =0 of the inner bend of the 1st dogleg.

$\theta_1 = -\theta_2$ in the dogleg and θ_1 in the dogleg is opposite to the bend in the Q-B half..

$T = -1/\theta_1 l_2 = 1/\theta' l_2'$ in dogleg and Q-B, so the product $\theta_1 l_2$ is fixed but not the individual θ or l. That is the bend of the Q-B can be different from those in the dogleg if the l_2's are chosen appropiately.

Now consider the order bend, then quad for 1st half of transform.

$$\begin{pmatrix} 1+lq+q_B(l+l_2+qll_2) & l+l_2+l_3+ql(l_2+l_3)+q_B l_3(l+l_2+qll_2) & 0 & \theta(l+l_2+qll_2) \\ q+q_B(1+ql_2) & 1+q(l_2+l_3)+q_B(l_3+ql_2 l_3) & 0 & \theta(1+ql_2) \\ -\theta & -\theta l_3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This wants to be of the form

$$\begin{pmatrix} A & B & 0 & \partial x/\partial z' \\ C & D & 0 & 0 \\ C\partial x/\partial z' & D\partial x/\partial z' & 1 & \partial z/\partial z' \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So again want $(1+ql_2) = 0$.

We see that with this geomentry $\partial z/\partial z' = 0$ because there is no second bend to develop an R56 path length difference..

The matrix becomes

$$\begin{pmatrix} 1+ql+q_B l_2 & l+l_3+l_3(ql+q_B l_2) & 0 & \theta l_2 \\ q & ql_3 & 0 & 0 \\ -\theta & -\theta l_3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The E and F terms relative to C, D and disp check correctly.

The **rules** for this 1st half matrix are:

$(1+ql_2)=0$

$T=-1/\theta l_2 = 1/\theta' l_2'$

The bend of a 2nd half Q-B must be opposite the 1st bend and can be scaled in inverse proportion to the lengths l_2, l_2'

For the Bend-Q-cav-Q-Bend soln the matrixes are

$$\begin{pmatrix} l_3 q & l_2+l_3+l_3(ql+q_B l_2) & 0 & \theta_2 l_3 \\ q & 1+ql+q_B l_2 & 0 & \theta \\ 0 & -\theta_2 l_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathrm{T} \begin{pmatrix} 1+ql+q_B l_2 & l+l_3+l_3(ql+q_B l_2) & 0 & \theta_1 l_2 \\ q & ql_3 & 0 & 0 \\ -\theta_1 & -\theta_1 l_3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

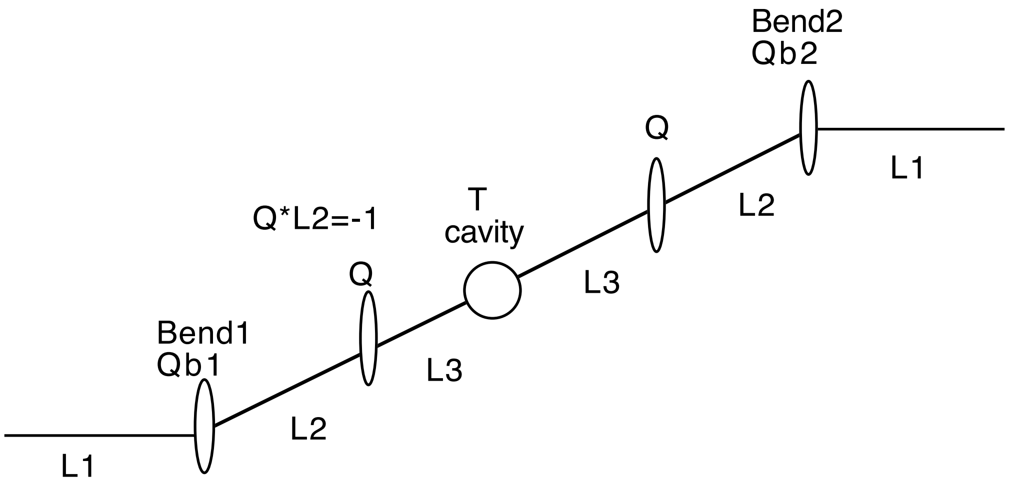
For a transform $[B \triangleright Q] \triangleright [T] \triangleright [Dogleg]$

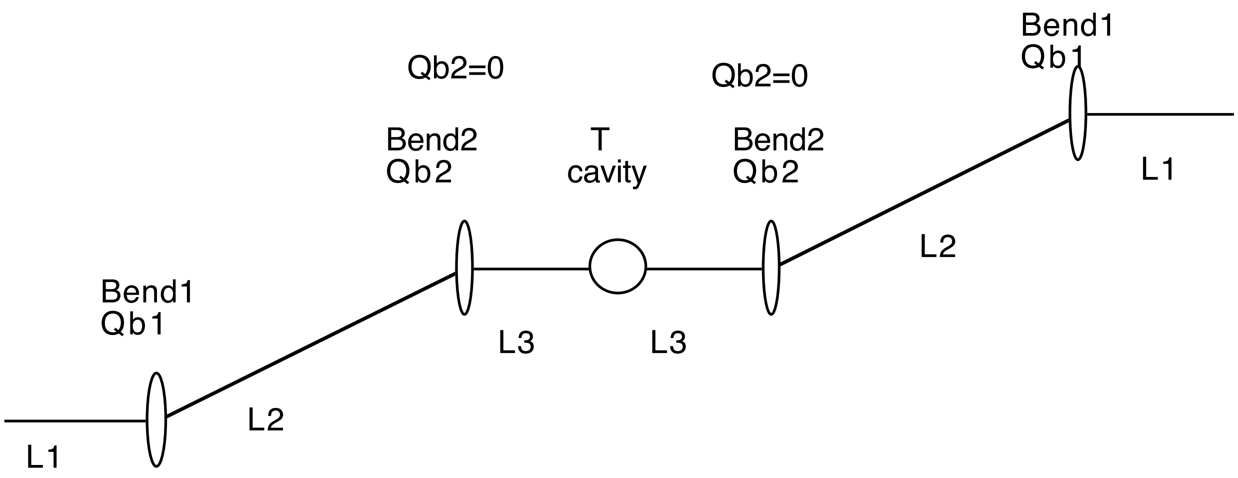
The **rules** become:

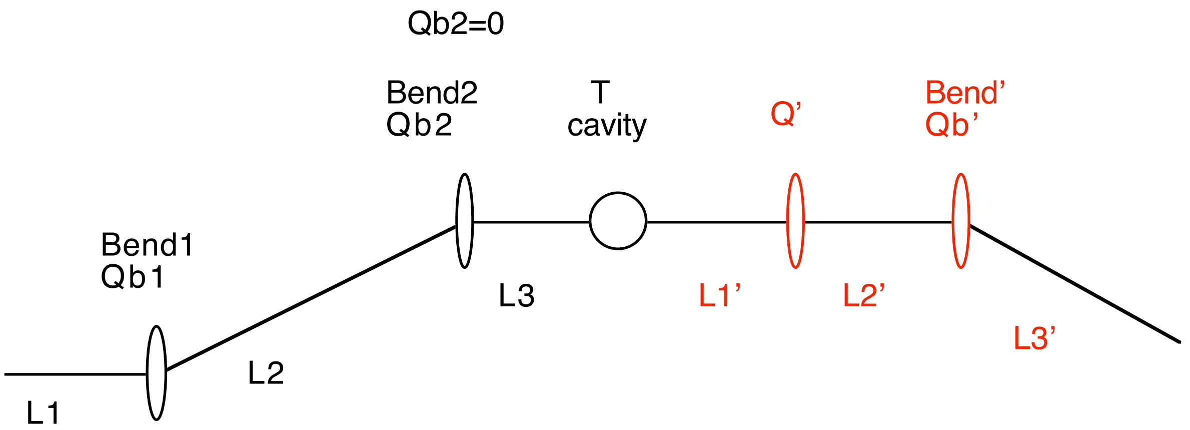
$(1+ql_2)=0$ for the quad in the 1st half [B>Q] matrix

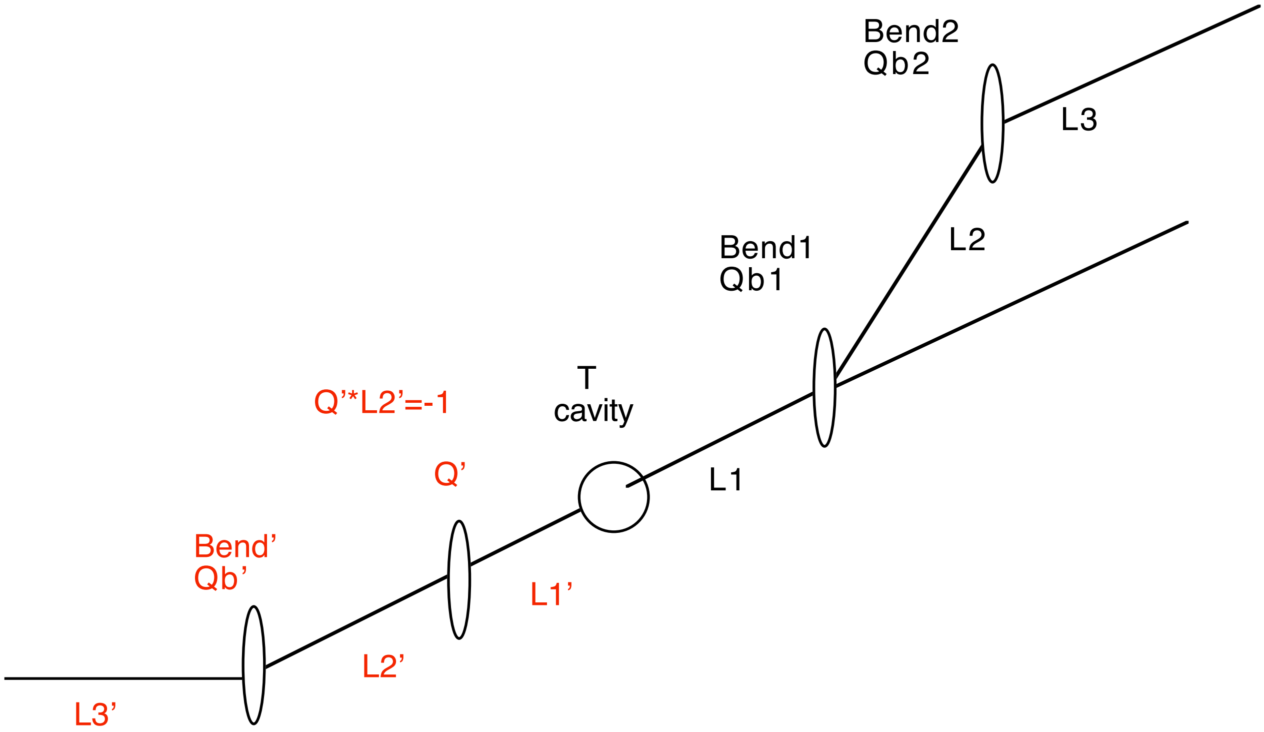
$T=-1/\theta l_2$ in the 1st half [B>Q] matrix, and $-1/T=\theta_1 l_2 = \partial z/\partial x'$ in the 2nd half [dogleg] matrix.That says the 1st bend of the dogleg is the **same** as for the bend of the [BQ]. **(This should be checked)**

q_B =0 of the inner (nearest cavity)bend of the dogleg..




Bend1
Qb1
Qb2=0
Qb2=0
Bend2
Qb2
T
cavity
Bend2
Qb2
L1
L2
Bend1
Qb1
L3
L3
L2
L1


Qb2=0
Bend2
Qb2
T
cavity
Q'
Bend'
Qb'
Bend1
Qb1
L3
L1'
L2'
L3'
L2
L1


Bend2
Qb2
L3
Bend1
Qb1
L2
T
cavity
Q'*L2'=-1
L1
Q'
Bend'
Qb'
L1'
L2'
L3'