An asteroid, headed directly toward Earth, has a speed of 12 km/s relative to the planet when the asteroid is 10 Earth radii from Earth’s center. Neglecting the effects of Earth's atmosphere on the asteroid, find the asteroid’s speed \( v_f \) when it reaches Earth’s surface.

**Calculations:** Let \( m \) represent the asteroid’s mass and \( M \) represent Earth’s mass \((5.98 \times 10^{24} \text{ kg})\). The asteroid is initially at distance \( 10R_E \) and finally at distance \( R_E \), where \( R_E \) is Earth’s radius \((6.37 \times 10^6 \text{ m})\). Substituting Eq. 13-21 for \( U \) and \( \frac{1}{2}mv^2 \) for \( K \), we rewrite Eq. 13-29 as

\[
\frac{1}{2}mv_f^2 - \frac{GMm}{R_E} = \frac{1}{2}mv_i^2 - \frac{GMm}{10R_E}
\]

Rearranging and substituting known values, we find

\[
v_f^2 = v_i^2 + \frac{2GM}{R_E} \left(1 - \frac{1}{10}\right)
\]

\[
= (12 \times 10^3 \text{ m/s})^2 + \frac{2(6.67 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{s}^2)(5.98 \times 10^{24} \text{ kg})}{6.37 \times 10^6 \text{ m}} 0.9
\]

\[
= 2.567 \times 10^8 \text{ m}^2/\text{s}^2,
\]

and

\[
v_f = 1.60 \times 10^4 \text{ m/s} = 16 \text{ km/s}. \quad (\text{Answer})
\]

At this speed, the asteroid would not have to be particularly large to do considerable damage at impact. If it were only 5 m across, the impact could release about as much energy as the nuclear explosion at Hiroshima. Alarming, about 500 million asteroids of this size are near Earth’s orbit, and in 1994 one of them apparently penetrated Earth’s atmosphere and exploded 20 km above the South Pacific (setting off nuclear-explosion warnings on six military satellites). The impact of an asteroid 500 m across (there may be a million of them near Earth’s orbit) could end modern civilization and almost eliminate humans worldwide.

**Additional examples, video, and practice available at WileyPLUS**
The orbit in Fig. 13-12 is described by giving its semimajor axis \( a \) and its eccentricity \( e \), the latter defined so that \( ea \) is the distance from the center of the ellipse to either focus \( F \) or \( F' \). An eccentricity of zero corresponds to a circle, in which the two foci merge to a single central point. The eccentricities of the planetary orbits are not large; so if the orbits are drawn to scale, they look circular. The eccentricity of the ellipse of Fig. 13-12, which has been exaggerated for clarity, is 0.74. The eccentricity of Earth’s orbit is only 0.0167.

2. THE LAW OF AREAS: A line that connects a planet to the Sun sweeps out equal areas in the plane of the planet’s orbit in equal time intervals; that is, the rate \( \frac{dA}{dt} \) at which it sweeps out area \( A \) is constant.

Qualitatively, this second law tells us that the planet will move most slowly when it is farthest from the Sun and most rapidly when it is nearest to the Sun. As it turns out, Kepler’s second law is totally equivalent to the law of conservation of angular momentum. Let us prove it.

The area of the shaded wedge in Fig. 13-13a closely approximates the area swept out in time \( \Delta t \) by a line connecting the Sun and the planet, which are separated by distance \( r \). The area \( \Delta A \) of the wedge is approximately the area of a triangle with base \( r \Delta \theta \) and height \( r \). Since the area of a triangle is one-half of the base times the height, \( \Delta A = \frac{1}{2} r^2 \Delta \theta \). This expression for \( \Delta A \) becomes more exact as \( \Delta \theta \) approaches zero. The instantaneous rate at which area is being swept out is then

\[
\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \omega, \tag{13-30}
\]

in which \( \omega \) is the angular speed of the rotating line connecting Sun and planet.

Figure 13-13b shows the linear momentum \( \vec{p} \) of the planet, along with the radial and perpendicular components of \( \vec{p} \). From Eq. 11-20 \((L = rp)\), the magnitude of the angular momentum \( L \) of the planet about the Sun is given by the product of \( r \) and \( p_\perp \), the component of \( \vec{p} \) perpendicular to \( r \). Here, for a planet of mass \( m \),

\[
L = rp_\perp = (r)(mv_\perp) = (r)(m\omega) = mr^2\omega, \tag{13-31}
\]

where we have replaced \( v_\perp \) with its equivalent \( \omega r \) (Eq. 10-18). Eliminating \( r^2\omega \) between Eqs. 13-30 and 13-31 leads to

\[
\frac{dA}{dt} = \frac{L}{2m}. \tag{13-32}
\]

If \( \frac{dA}{dt} \) is constant, as Kepler said it is, then Eq. 13-32 means that \( L \) must also be constant—angular momentum is conserved. Kepler’s second law is indeed equivalent to the law of conservation of angular momentum.
CHAPTER 13 GRAVITATION

Fig. 13-14 A planet of mass \( m \) moving around the Sun in a circular orbit of radius \( r \).

**Table 13-3**

<table>
<thead>
<tr>
<th>Planet</th>
<th>Semimajor Axis ( a ) (10(^{10}) m)</th>
<th>Period ( T ) (y)</th>
<th>( T^2/a^3 ) (10(^{-34}) y(^2)m(^3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>5.79</td>
<td>0.241</td>
<td>2.99</td>
</tr>
<tr>
<td>Venus</td>
<td>10.8</td>
<td>0.615</td>
<td>3.00</td>
</tr>
<tr>
<td>Earth</td>
<td>15.0</td>
<td>1.00</td>
<td>2.96</td>
</tr>
<tr>
<td>Mars</td>
<td>22.8</td>
<td>1.88</td>
<td>2.98</td>
</tr>
<tr>
<td>Jupiter</td>
<td>77.8</td>
<td>11.9</td>
<td>3.01</td>
</tr>
<tr>
<td>Saturn</td>
<td>143</td>
<td>29.5</td>
<td>2.98</td>
</tr>
<tr>
<td>Uranus</td>
<td>287</td>
<td>84.0</td>
<td>2.98</td>
</tr>
<tr>
<td>Neptune</td>
<td>450</td>
<td>165</td>
<td>2.99</td>
</tr>
<tr>
<td>Pluto</td>
<td>590</td>
<td>248</td>
<td>2.99</td>
</tr>
</tbody>
</table>

**3. THE LAW OF PERIODS:** The square of the period of any planet is proportional to the cube of the semimajor axis of its orbit.

To see this, consider the circular orbit of Fig. 13-14, with radius \( r \) (the radius of a circle is equivalent to the semimajor axis of an ellipse). Applying Newton’s second law \( (F = ma) \) to the orbiting planet in Fig. 13-14 yields

\[
\frac{GMm}{r^2} = (m)(\omega^2 r).
\]

Here we have substituted from Eq. 13-1 for the force magnitude \( F \) and used Eq. 10-23 to substitute \( \omega r \) for the centripetal acceleration. If we now use Eq. 10-20 to replace \( \omega \) with \( 2\pi T \), where \( T \) is the period of the motion, we obtain Kepler’s third law:

\[
T^2 = \left(\frac{4\pi^2}{GM}\right)a^3 \quad \text{(law of periods).}
\]

The quantity in parentheses is a constant that depends only on the mass \( M \) of the central body about which the planet orbits.

Equation 13-34 holds also for elliptical orbits, provided we replace \( r \) with \( a \), the semimajor axis of the ellipse. This law predicts that the ratio \( T^2/a^3 \) has essentially the same value for every planetary orbit around a given massive body. Table 13-3 shows how well it holds for the orbits of the planets of the solar system.

**CHECKPOINT 4**

Satellite 1 is in a certain circular orbit around a planet, while satellite 2 is in a larger circular orbit. Which satellite has (a) the longer period and (b) the greater speed?

**Sample Problem**

Kepler’s law of periods, Comet Halley

Comet Halley orbits the Sun with a period of 76 years and, in 1986, had a distance of closest approach to the Sun, its perihelion distance \( R_p \), of \( 8.9 \times 10^{10} \) m. Table 13-3 shows that this is between the orbits of Mercury and Venus.

(a) What is the comet’s farthest distance from the Sun, which is called its aphelion distance \( R_a \)?

**KEY IDEAS**

From Fig. 13-12, we see that \( R_a + R_p = 2a \), where \( a \) is the semimajor axis of the orbit. Thus, we can find \( R_a \) if we first find \( a \). We can relate \( a \) to the given period via the law of periods (Eq. 13-34) if we simply substitute the semimajor axis \( a \) for \( r \).

**Calculations:** Making that substitution and then solving for \( a \), we have

\[
a = \left(\frac{GMT^2}{4\pi^2}\right)^{1/3}.
\]  

(13-35)

If we substitute the mass \( M \) of the Sun, \( 1.99 \times 10^{30} \) kg, and the period \( T \) of the comet, 76 years or \( 2.4 \times 10^8 \) s, into Eq. 13-35, we find that \( a = 2.7 \times 10^{12} \) m. Now we have

\[
R_a = 2a - R_p = (2)(2.7 \times 10^{12} \text{ m}) - 8.9 \times 10^{10} \text{ m} = 5.3 \times 10^{12} \text{ m}.
\]  

(Answer)

Table 13-3 shows that this is a little less than the semimajor axis of the orbit of Pluto. Thus, the comet does not get farther from the Sun than Pluto.

(b) What is the eccentricity \( e \) of the orbit of comet Halley?

**KEY IDEA**

We can relate \( e, a, \) and \( R_p \) via Fig. 13-12, in which we see that \( ea = a - R_p \).

**Calculation:** We have

\[
e = \frac{a - R_p}{a} = 1 - \frac{R_p}{a} \quad \text{(13-36)}
\]

\[
e = 1 - \frac{8.9 \times 10^{10} \text{ m}}{2.7 \times 10^{12} \text{ m}} = 0.97. \quad \text{(Answer)}
\]

This tells us that, with an eccentricity approaching unity, this orbit must be a long thin ellipse.
13-8 Satellites: Orbits and Energy

As a satellite orbits Earth in an elliptical path, both its speed, which fixes its kinetic energy $K$, and its distance from the center of Earth, which fixes its gravitational potential energy $U$, fluctuate with fixed periods. However, the mechanical energy $E$ of the satellite remains constant. (Since the satellite’s mass is so much smaller than Earth’s mass, we assign $U$ and $E$ for the Earth–satellite system to the satellite alone.)

The potential energy of the system is given by Eq. 13-21:

$$U = -\frac{GMm}{r}$$

(with $U = 0$ for infinite separation). Here $r$ is the radius of the satellite’s orbit, assumed for the time being to be circular, and $M$ and $m$ are the masses of Earth and the satellite, respectively.

To find the kinetic energy of a satellite in a circular orbit, we write Newton’s second law ($F = ma$) as

$$\frac{GMm}{r^2} = m\frac{v^2}{r},$$

(13-37)

where $v^2/r$ is the centripetal acceleration of the satellite. Then, from Eq. 13-37, the kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{GMm}{2r},$$

(13-38)

which shows us that for a satellite in a circular orbit,

$$K = -\frac{U}{2} \quad \text{(circular orbit)}.$$  

(13-39)

The total mechanical energy of the orbiting satellite is

$$E = K + U = \frac{GMm}{2r} - \frac{GMm}{r}$$

or

$$E = -\frac{GMm}{2r} \quad \text{(circular orbit)}.$$  

(13-40)

This tells us that for a satellite in a circular orbit, the total energy $E$ is the negative of the kinetic energy $K$:

$$E = -K \quad \text{(circular orbit)}.$$  

(13-41)

For a satellite in an elliptical orbit of semimajor axis $a$, we can substitute $a$ for $r$ in Eq. 13-40 to find the mechanical energy:

$$E = -\frac{GMm}{2a} \quad \text{(elliptical orbit)}.$$  

(13-42)

Equation 13-42 tells us that the total energy of an orbiting satellite depends only on the semimajor axis of its orbit and not on its eccentricity $e$. For example, four orbits with the same semimajor axis are shown in Fig. 13-15; the same satellite would have the same total mechanical energy $E$ in all four orbits. Figure 13-16 shows the variation of $K$, $U$, and $E$ with $r$ for a satellite moving in a circular orbit about a massive central body.

**Fig. 13-15** Four orbits with different eccentricities $e$ about an object of mass $M$. All four orbits have the same semimajor axis $a$ and thus correspond to the same total mechanical energy $E$.

**Fig. 13-16** The variation of kinetic energy $K$, potential energy $U$, and total energy $E$ with radius $r$ for a satellite in a circular orbit. For any value of $r$, the values of $U$ and $E$ are negative, the value of $K$ is positive, and $E = -K$. As $r \to \infty$, all three energy curves approach a value of zero.
CHAPTER 13 GRAVITATION

Sample Problem

Mechanical energy of orbiting bowling ball

A playful astronaut releases a bowling ball, of mass \( m = 7.20 \text{ kg} \), into circular orbit about Earth at an altitude \( h \) of 350 km.

(a) What is the mechanical energy \( E \) of the ball in its orbit?

**KEY IDEA**

We can get \( E \) from the orbital energy, given by Eq. 13-40 \((E = -\frac{GMm}{2r})\), if we first find the orbital radius \( r \). (It is not simply the given altitude.)

**Calculations:** The orbital radius must be

\[
r = R + h = 6370 \text{ km} + 350 \text{ km} = 6.72 \times 10^6 \text{ m},
\]

in which \( R \) is the radius of Earth. Then, from Eq. 13-40, the mechanical energy is

\[
E = \frac{-GMm}{2r} = \frac{- (6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)(5.98 \times 10^{24} \text{ kg})(7.20 \text{ kg})}{2(6.72 \times 10^6 \text{ m})} = -2.14 \times 10^8 \text{ J} = -214 \text{ MJ}.
\]

(b) What is the mechanical energy \( E_0 \) of the ball on the launchpad at Cape Canaveral (before it, the astronaut, and the spacecraft are launched)? From there to the orbit, what is the change \( \Delta E \) in the ball’s mechanical energy?

**KEY IDEA**

On the launchpad, the ball is not in orbit and thus Eq. 13-40 does not apply. Instead, we must find \( E_0 = K_0 + U_0 \), where \( K_0 \) is the ball's kinetic energy and \( U_0 \) is the gravitational potential energy of the ball–Earth system.

**Calculations:** To find \( U_0 \), we use Eq. 13-21 to write

\[
U_0 = -\frac{GMm}{R} = \frac{- (6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)(5.98 \times 10^{24} \text{ kg})(7.20 \text{ kg})}{6.37 \times 10^8 \text{ m}} = -4.51 \times 10^8 \text{ J} = -451 \text{ MJ}.
\]

The kinetic energy \( K_0 \) of the ball is due to the ball’s motion with Earth’s rotation. You can show that \( K_0 \) is less than 1 MJ, which is negligible relative to \( U_0 \). Thus, the mechanical energy of the ball on the launchpad is

\[
E_0 = K_0 + U_0 = 0 - 451 \text{ MJ} = -451 \text{ MJ}.
\]

The *increase* in the mechanical energy of the ball from launchpad to orbit is

\[
\Delta E = E - E_0 = (-214 \text{ MJ}) - (-451 \text{ MJ}) = 237 \text{ MJ}.
\]

This is worth a few dollars at your utility company. Obviously the high cost of placing objects into orbit is not due to their required mechanical energy.