Part I
Cherenkov radiation.

Review:

1. Maxwell equations in media. Frequency dependence of the dielectric constant \( \epsilon(\omega) \).

2. Relation between the dissipation and imaginary part of the response (for example Landau and Lifshitz, vol V (Statistical Physics), section 123.

1 Energy-momentum conservation and general arguments.

Electron (or another charged particle) emits photons with energy \( E = \hbar \omega \) at the angle \( \theta \) to its direction.

As we shall see below, this emission is possible only if the electron velocity \( v > c' \) is larger than the velocity of the light in the medium. This, in turn is possible, only for \( \epsilon(\omega) > 1 \) which is due to media. At very high frequencies \( \epsilon \rightarrow 1 \) making the emission impossible. Therefore, the frequencies at which emission is possible are of the order of atomic frequencies \( (1eV - 10eV) \) which are much smaller than electron energy \( E > mc^2 = 0.5MeV \).

If electron with momentum \( p \) and energy \( E \) emits a photon with \( \Delta E = c' \) \( \Delta p \) its energy and momentum becomes \( E - \Delta E \) and \( (\vec{p} - \Delta \vec{p}) \) with \( \Delta E = \vec{v} \Delta p = v \cos \theta (\Delta E/c') \). We conclude that electron emits a photon at the angle \( \cos \theta = c'/v \). In each emission process its energy changes by \( \Delta E = v \Delta p \) which is carried away by the photon \( \Delta E = c'|\Delta p| \).

These arguments do not give the power of radiation but they show the direction of the light and give the condition when such process is possible.

2 Radiated power.

To compute the radiated power we need to solve Maxwell equations. There are options: we may compute the energy carried away by electromagnetic field at large distances from the particle in integrate over all angles or we may compute the back reaction of the field that slows down the particle. Both should give the same results due to energy conservation. The second is slightly easier and we shall use it.

The energy of the particle changes due to electric field. Power (energy per time) \( P = -e \vec{v} \vec{E} \). Here \( E \) is the electric field at the position of the particle \( r = vt \). Note that we do not take into account the slowing down of the particle in this approximation: attempt to ‘improve’ this calculation making it self-consistent would lead to divergencies and other disasters.
Thus we need to compute the electric field \( E(r = vt) \) created by particle with charge \( e \) moving as \( r = vt \), i.e. by charge and current densities

\[
\rho = e \delta(\overrightarrow{r} - \overrightarrow{vt}) \\
j = e \overrightarrow{v} \delta(\overrightarrow{r} - \overrightarrow{vt})
\]

which Fourier transforms \( (\rho(\omega, k) = \int \rho(t, r) \exp(i\omega t - i \overrightarrow{k} \overrightarrow{r}) dt d^3r, j(\omega, k) = \int j(t, r) \exp(i\omega t - i \overrightarrow{k} \overrightarrow{r}) dt d^3r) \) are

\[
\rho = 2\pi e \delta(\omega - \overrightarrow{v} \overrightarrow{k}) \\
j = 2\pi e \overrightarrow{v} \delta(\omega - \overrightarrow{v} \overrightarrow{k})
\]

To simplify the algebra we assume \( \mu = 1 \) (but \( \epsilon \neq 1 \)) and use Gauss units. Introducing the vector and scalar potential as usual and taking the Fourier transforms we get the Maxwell equations for the potentials

\[
(k^2 - \epsilon(\omega)\omega^2/c^2)A = \frac{4\pi}{c} j \\
(k^2 - \epsilon(\omega)\omega^2/c^2)\phi = \frac{4\pi}{\epsilon(\omega)} j
\]

Using the expression for \( \rho \) and \( j \) found above we get

\[
A = \frac{4\pi}{c} \frac{2\pi e \overrightarrow{v} \delta(\omega - \overrightarrow{v} \overrightarrow{k})}{(k^2 - \epsilon(\omega)\omega^2/c^2)} \\
\phi = \frac{4\pi}{\epsilon} \frac{2\pi e \overrightarrow{v} \delta(\omega - \overrightarrow{v} \overrightarrow{k})}{(k^2 - \epsilon(\omega)\omega^2/c^2)}
\]

Electric field is related to the vector and scalar potential by \( E = -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \phi \) which after Fourier transformation becomes \( E(\omega, k) = i(\omega/c \overrightarrow{A} - k \phi) \). Substituting in this equation the expressions for \( A \) and \( \phi \) we get the Fourier components of the electric field

\[
\overrightarrow{E}(k, \omega) = 4\pi i e \left( \frac{\omega}{c^2} \overrightarrow{v} - \overrightarrow{k} \frac{1}{\epsilon} \right) \frac{2\pi e \delta(\omega - \overrightarrow{v} \overrightarrow{k})}{(k^2 - \epsilon(\omega)\omega^2/c^2)}
\]

We need to find the \( e \overrightarrow{v} \overrightarrow{E}(r = vt) = \int e \overrightarrow{v} \overrightarrow{E}(k, \omega) \exp(-i\omega t + i \overrightarrow{v} \overrightarrow{k} t) d^3k d\omega/(2\pi)^3 \). Using the above expression for \( E(k, \omega) \) in this formula and integrating over frequencies which removes the \( \delta \)–function we get for the dissipated power

\[
P = -4\pi i e^2 \int \left( \frac{\overrightarrow{v}^2}{c^2} - \frac{1}{\epsilon(\overrightarrow{v} \overrightarrow{k})} \right) \frac{\overrightarrow{v} \overrightarrow{k}}{k^2 - \epsilon(\overrightarrow{v} \overrightarrow{k})^2/c^2} \frac{d^3k}{(2\pi)^3}
\]

Physical dissipated power corresponds to the real part of this expression so we get
\[ P = 4\pi e^2 Im \int \left( \frac{v^2}{c^2} - \frac{1}{\epsilon(\overrightarrow{v} \overrightarrow{k})} \right) \frac{\overrightarrow{v} \overrightarrow{k}}{k^2 - \epsilon(\overrightarrow{v} \overrightarrow{k})^2/c^2} \frac{d^3k}{(2\pi)^3} \]

Integration over three dimensional \( k \) is convenient to separate into two dimensional \( q \) perpendicular to \( \overrightarrow{v} \) and one dimensional \( \omega = \overrightarrow{v} \overrightarrow{k} \). In the denominator \( k^2 = q^2 + \omega^2/v^2 \). Additional convenience of this choice of variables is that \( \omega \) has a meaning of the frequency of the emitted wave due to \( \delta \)–functions in previous equations. Thus, it acquires an infinitesimal imaginary part at the poles:

\[ P = e^2 Im \int \left( \frac{v^2}{c^2} - \frac{1}{\epsilon(\omega)} \right) \frac{2\omega}{q^2 + (\omega + i0)^2(1/v^2 - \epsilon(\omega)/c^2)} \frac{d^2qd\omega}{v(2\pi)^2} \]

To simplify this expression we note that for real (non-dissipative) \( \epsilon(\omega) \) it is even function of frequency: \( \epsilon(\omega) = \epsilon(-\omega) \). Imaginary part of

\[ \frac{Im}{q^2 + A(\omega + i0)^2} \]

is also even function of \( \omega \), so the integral over frequencies can be performed from 0 to \( \infty \) and the result multiplied by 2. Further, the integral over momenta \( d^2q \) is simplified in cylindrical coordinates (nothing depends on the angle):

\[ P = e^2 Im \int_0^\infty d\omega \int \left( \frac{v^2}{c^2} - \frac{1}{\epsilon(\omega)} \right) \frac{2\omega}{q^2 + (\omega + i0)^2(1/v^2 - \epsilon(\omega)/c^2)} \frac{dq}{\pi} \]

Finally the integral over \( dq \) is

\[ \frac{1}{2} dq^2: \]

\[ Im \int_0^\infty \frac{dx}{x + (\omega + i0)^2(1/v^2 - \epsilon(\omega)/c^2)} = \pi \Theta(1 - \frac{c^2}{\epsilon(\omega)v^2}) \]

where \( \Theta \)-function represents the condition that imaginary part appears only if \( x = q^2 = \omega^2(\epsilon(\omega)/c^2 - 1/v^2) \) for some positive \( x \). We come to the final result

\[ P = ve^2 Im \int_0^\infty d\omega(1 - \frac{c^2}{\epsilon(\omega)v^2})\Theta(1 - \frac{c^2}{\epsilon(\omega)v^2}) \]