2.23 A hollow cube has conducting walls defined by six planes $x = 0$, $y = 0$, $z = 0$, and $x = a$, $y = a$, $z = a$. The walls $z = 0$ and $z = a$ are held at a constant potential $V$. The other four sides are at zero potential.

a) Find the potential $\Phi(x, y, z)$ at any point inside the cube.

The potential may be obtained by superposition

$$\Phi = \Phi_{\text{top}} + \Phi_{\text{bottom}}$$

where $\Phi_{\text{top}}$ ($\Phi_{\text{bottom}}$) is the solution for a hollow cube with the top (bottom) held at constant potential $V$ and all other sides at zero potential. As we have seen, the series solution for $\Phi_{\text{top}}$ is given by

$$\Phi_{\text{top}} = \sum_{n,m} A_{n,m} \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{m\pi x}{a} \right) \sinh \left( \frac{\sqrt{n^2 + m^2} \pi z}{a} \right)$$

where

$$A_{n,m} = \frac{4}{a^2 \sinh(\sqrt{n^2 + m^2} \pi)} \int_0^a dx \int_0^a dy \int_0^a V \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{m\pi x}{a} \right)$$
Noting that
\[ \int_0^a \sin \left( \frac{n\pi x}{a} \right) dx = -\frac{a}{n\pi} \cos \left( \frac{n\pi x}{a} \right) \bigg|_0^a = \frac{a}{n\pi} (1 - (-1)^n) = \frac{2a}{n\pi} \text{ for } n \text{ odd} \]
we have
\[ A_{n,m} = \frac{16V}{nm\pi^2 \sinh(\sqrt{n^2 + m^2} \pi)} \quad n, m \text{ odd} \]
and hence
\[ \Phi_{\text{top}} = \frac{16V}{\pi^2} \sum_{n,m \text{ odd}} \frac{1}{nm \sinh(\sqrt{n^2 + m^2} \pi)} \]
\[ \times \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{m\pi x}{a} \right) \sinh \left( \frac{\sqrt{n^2 + m^2} \pi z}{a} \right) \]
To obtain \( \Phi_{\text{bottom}} \), it is sufficient to realize that symmetry allows us to take \( z \to a - z \). More precisely
\[ \Phi_{\text{bottom}}(x, y, z) = \Phi_{\text{top}}(x, y, a - z) \]
As a result
\[ \Phi = \Phi_{\text{top}} + \Phi_{\text{bottom}} \]
\[ = \frac{16V}{\pi^2} \sum_{n,m \text{ odd}} \frac{1}{nm \sinh(\sqrt{n^2 + m^2} \pi)} \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{m\pi x}{a} \right) \]
\[ \times \left[ \sinh \left( \frac{\sqrt{n^2 + m^2} \pi z}{a} \right) + \sinh \left( \frac{\sqrt{n^2 + m^2} \pi(a - z)}{a} \right) \right] \]
Note that this may be simplified using
\[ \sinh \zeta + \sinh(\alpha - \zeta) = 2 \sinh(\alpha/2) \cosh(\zeta - \alpha/2) \]
to read
\[ \Phi = \frac{16V}{\pi^2} \sum_{n,m \text{ odd}} \frac{1}{nm \cosh(\sqrt{n^2 + m^2} \pi/2)} \]
\[ \times \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{m\pi x}{a} \right) \cosh \left( \frac{\sqrt{n^2 + m^2} \pi(z - a/2)}{a} \right) \]
(17)
b) Evaluate the potential at the center of the cube numerically, accurate to three significant figures. How many terms in the series is it necessary to keep in order to attain this accuracy? Compare your numerical result with the average value of the potential on the walls. See Problem 2.28.
At the center of the cube, \((x, y, z) = (a/2, a/2, a/2)\), the potential from (17) reads

\[
\Phi(\text{center}) = \frac{16V}{\pi^2} \sum_{n,m \text{ odd}} \frac{\sin(n\pi/2) \sin(m\pi/2)}{nm \cosh(\sqrt{n^2 + m^2} \pi/2)}
\]

\[
= \frac{16V}{\pi^2} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{(2i+1)(2j+1) \cosh(\sqrt{(2i+1)^2 + (2j+1)^2} \pi/2)}
\]

Numerically, the first few terms in this series are given by

<table>
<thead>
<tr>
<th>(n)</th>
<th>(m)</th>
<th>(\Phi_{n,m}/V)</th>
<th>running total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>.347546</td>
<td>.347546</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>-.007524</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-.007524</td>
<td>.332498</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>.000460</td>
<td>.332958</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>.000215</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>.000215</td>
<td>.333389</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>-.000023</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>-.000023</td>
<td>.333343</td>
</tr>
</tbody>
</table>

This table indicates that we need to keep at least the first four terms to achieve accuracy to three significant figures. To this level of accuracy, we have

\[
\Phi(\text{center}) \approx .333V
\]

If we went to higher orders, it appears that the potential at the center is precisely

\[
\Phi(\text{center}) = \frac{1}{3}V
\]

which is the average value of the potential on the walls. In fact, we can prove (as in Problem 2.28) that the potential at the center of a regular polyhedron is equal to the average of the potential on the walls. Hence this value of \(V/3\) is indeed exact.

c) Find the surface-charge density on the surface \(z = a\).

For the surface-charge density on the inside top surface \((z = a)\), we use

\[
\sigma = -\varepsilon_0 \frac{\partial \Phi}{\partial n} \bigg|_S = \varepsilon_0 \frac{\partial \Phi}{\partial z} \bigg|_{z=a}
\]

where the normal pointing away from the top conductor is \(\hat{n} = -\hat{z}\). This is what accounts for the sign flip in the above. Substituting in (17) gives

\[
\sigma = \frac{16\varepsilon_0 V}{\pi a} \sum_{n,m \text{ odd}} \frac{\sqrt{n^2 + m^2}}{nm} \tanh(\sqrt{n^2 + m^2} \pi/2) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right)
\]
3.1 Two concentric spheres have radii $a$, $b$ ($b > a$) and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential $V$. The other hemispheres are at zero potential.

Determine the potential in the region $a \leq r \leq b$ as a series in Legendre polynomials. Include terms at least up to $l = 4$. Check your solution against known results in the limiting cases $b \to \infty$, and $a \to 0$.

The general expansion in Legendre polynomials is of the form

$$
\Phi(r, \theta) = \sum_{l} [A_l r^l + B_l r^{-l-1}]P_l(\cos \theta)
$$

(1)

Since we are working in the region between spheres, neither $A_l$ nor $B_l$ can be assumed to vanish. To solve for both $A_l$ and $B_l$ we will need to consider boundary conditions at $r = a$ and $r = b$

$$
\Phi(a, \theta) = \sum_{l} [A_l a^l + B_l a^{-l-1}]P_l(\cos \theta) = \begin{cases} V & \cos \theta \geq 0 \\ 0 & \cos \theta < 0 \end{cases}
$$

$$
\Phi(b, \theta) = \sum_{l} [A_l b^l + B_l b^{-l-1}]P_l(\cos \theta) = \begin{cases} 0 & \cos \theta > 0 \\ V & \cos \theta \leq 0 \end{cases}
$$

Using orthogonality of the Legendre polynomials, we may write

$$
A_l a^l + B_l a^{-l-1} = \frac{2l+1}{2} V \int_{0}^{1} P_l(x) \, dx
$$

$$
A_l b^l + B_l b^{-l-1} = \frac{2l+1}{2} V \int_{-1}^{0} P_l(x) \, dx = \frac{2l+1}{2} V (-1)^l \int_{0}^{1} P_l(x) \, dx
$$

where in the last expression we used the fact that $P_l(-x) = (-1)^l P_l(x)$. Since the integral is only over half of the standard interval, it does not yield a particularly simple result. For now, we define

$$
N_l = \int_{0}^{1} P_l(x) \, dx
$$

(2)

As a result, we have the system of equations

$$
\begin{pmatrix}
A_l \\
B_l
\end{pmatrix} = \frac{2l+1}{2} V N_l \begin{pmatrix}
1 \\
(-1)^l
\end{pmatrix}
$$
which may be solved to give
\[
\begin{pmatrix}
  A_l \\
  B_l
\end{pmatrix}
= \frac{2l+1}{2} VN_l \frac{1}{b^{2l+1} - a^{2l+1}} \left( \frac{(-1)^l b^{l+1} - a^{l+1}}{(ab)^{l+1}(b^l + (-1)^{l+1}a^l)} \right)
\]
Inserting this into (1) gives
\[
\Phi(r, \theta) = \frac{1}{2} V \sum_l \frac{(2l+1)N_l}{1 - \left( \frac{a}{b} \right)^{2l+1}} \left[ (-1)^l \left( 1 + (-1)^l \left( \frac{a}{b} \right)^{l+1} \right) \left( \frac{r}{b} \right)^{l+1} \\
+ \left( 1 + (-1)^l \left( \frac{a}{b} \right)^{l+1} \right) \left( \frac{a}{r} \right)^{l+1} \right] P_l(\cos \theta)
\]
We now examine the integral (2). First note that for even $l$ we may actually extend the region of integration
\[
N_{2j} = \int_0^1 P_{2j}(x) \, dx = \frac{1}{2} \int_{-1}^1 P_{2j}(x) \, dx = \frac{1}{2} \int_{-1}^1 P_0(x)P_{2j}(x) \, dx = \delta_{j,0}
\]
This demonstrates that the only contribution from even $l$ is for $l = 0$, corresponding to the average potential. Using this fact, the potential (3) reduces to
\[
\Phi(r, \theta) = \frac{V}{2} + \frac{V}{2} \sum_{j=1}^\infty \frac{(4j-1)N_{2j-1}}{1 - \left( \frac{a}{b} \right)^{4j-1}} \left[ - \left( 1 + \left( \frac{a}{b} \right)^{2j} \right) \left( \frac{r}{b} \right)^{2j-1} \\
+ \left( 1 + \left( \frac{a}{b} \right)^{2j-1} \right) \left( \frac{a}{r} \right)^{2j} \right] P_{2j-1}(\cos \theta)
\]
Physically, once the average $V/2$ is removed, the remaining potential is odd under the flip $z \rightarrow -z$ or $\cos \theta \rightarrow -\cos \theta$. This is why only odd Legendre polynomials may contribute.
Note that an alternative method of solution would be to use linear superposition
\[
\Phi = \Phi_{\text{inner}} + \Phi_{\text{outer}}
\]
where $\Phi_{\text{inner}}$ is the solution where the inner sphere has potential $V_a(\theta)$ and the outer sphere is grounded, and where $\Phi_{\text{outer}}$ is the solution where the outer sphere has potential $V_b(\theta)$ and the inner sphere is grounded. To calculate $\Phi_{\text{inner}}$ we note that the boundary conditions are such that $\Phi_{\text{inner}}(r = b) = 0$. This motivates an expansion of the form
\[
\Phi_{\text{inner}}(r, \theta) = \sum_l \alpha_l \left( \frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) P_l(\cos \theta)
\]
The boundary condition at \( r = a \) is then

\[
V_a(\theta) = \sum_l \frac{\alpha_l}{a^{l+1}} (1 - (a/b)^{2l+1}) P_l(\cos \theta)
\]

which, by orthogonality, gives

\[
\alpha_l = \frac{2l + 1}{2} \frac{a^{l+1}}{1 - (a/b)^{2l+1}} \int_{-1}^{1} V_a(\cos \theta) P_l(\cos \theta) \, d(\cos \theta)
\]

Similarly, for \( \Phi_{\text{outer}} \), we may interchange \( a \leftrightarrow b \) and rearrange the expressions to obtain

\[
\Phi_{\text{outer}}(r, \theta) = \sum_l \beta_l \left( r^l - \frac{a^{2l+1}}{b^{l+1}} \right) P_l(\cos \theta)
\]

where

\[
\beta_l = \frac{2l + 1}{2} \frac{1/b^l}{1 - (a/b)^{2l+1}} \int_{-1}^{1} V_b(\cos \theta) P_l(\cos \theta) \, d(\cos \theta)
\]

Using \( V_a = V \) for \( \cos \theta > 0 \) and \( V_b = V \) for \( \cos \theta < 0 \) gives explicitly

\[
\Phi_{\text{inner}} = \sum_l \frac{2l + 1}{2} \frac{V N_l}{1 - (a/b)^{2l+1}} \left[ \left( \frac{a}{r} \right)^{l+1} - \left( \frac{a}{b} \right)^{l+1} \left( \frac{r}{b} \right)^{l} \right] P_l(\cos \theta)
\]

\[
\Phi_{\text{outer}} = \sum_l \frac{2l + 1}{2} \frac{(-1)^l V N_l}{1 - (a/b)^{2l+1}} \left[ \left( \frac{r}{b} \right)^{l} - \left( \frac{a}{b} \right)^{l} \left( \frac{a}{r} \right)^{l+1} \right] P_l(\cos \theta)
\]

When superposed, the solution is identical to (3) which we found above.

At this stage, we may simply perform elementary integrations to obtain the first few terms \( N_1, N_3, \) etc. However, we may derive a fairly simple expression for \( N_l \) by integrating the generating function

\[
(1 - 2xt + t^2)^{-1/2} = \sum_{l=0}^{\infty} P_l(x)t^l
\]

from \( x = 0 \) to 1. In other words

\[
\sum_{l=0}^{\infty} N_l t^l = \int_{0}^{1} (1 - 2xt + t^2)^{-1/2} \, dx = t^{-1}(-1 + t + \sqrt{1 + t^2})
\]

The square root yields a binomial expansion

\[
(1 + t^2)^{1/2} = 1 + \frac{1}{2} t^2 + \frac{1}{2} (-\frac{1}{2}) \frac{1}{2!} t^4 + \frac{1}{2} (-\frac{1}{2})(-\frac{3}{2}) \frac{1}{3!} t^6 + \cdots = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j}{\Gamma(\frac{j}{2})} \frac{1}{j!} t^{2j}
\]
As a result
\[ \sum_{l=0}^{\infty} N_l t^l = 1 + \sum_{j=1}^{\infty} (-j+1)^{\frac{j}{2}} - \frac{\Gamma(j - \frac{1}{2})}{2\sqrt{\pi}j!} t^{2j-1} \]
where we used the fact that $\Gamma(-\frac{1}{2}) = -2\Gamma(\frac{1}{2}) = -2\sqrt{\pi}$. Matching powers of $t$ demonstrates that all even $N_l$ terms vanish except $N_0 = 1$ and that
\[ N_{2j-1} = (-1)^{j+1} \frac{\Gamma(j - \frac{1}{2})}{2\sqrt{\pi}j!} \]

The final result for the potential is thus
\[ \Phi(r, \theta) = \frac{V}{2} + V \sum_{j=1}^{\infty} \frac{(-1)^{j+1}(4j - 1)\Gamma(j - \frac{1}{2})}{4\sqrt{\pi}j!(1 - \left(\frac{a}{b}\right)^{4j-1})} \left[ -\left(1 + \left(\frac{a}{b}\right)^{2j}\right) \left(\frac{r}{b}\right)^{2j-1} \right. \\
\left. + \left(1 + \left(\frac{a}{b}\right)^{2j-1}\right) \left(\frac{a}{r}\right)^{2j} \right] P_{2j-1}(\cos \theta) \]
\[ = \frac{V}{2} + V \left[ \frac{3}{4} \left(1 - \left(\frac{a}{b}\right)^3\right)^{-1} \left( -\left(1 + \left(\frac{a}{b}\right)^2\right) \left(\frac{r}{b}\right) + \left(1 + \left(\frac{a}{b}\right)\right) \left(\frac{a}{r}\right)^2 \right) P_1(\cos \theta) \right. \\
\left. - \frac{7}{16} \left(1 - \left(\frac{a}{b}\right)^7\right)^{-1} \left( -\left(1 + \left(\frac{a}{b}\right)^4\right) \left(\frac{r}{b}\right)^3 + \left(1 + \left(\frac{a}{b}\right)^3\right) \left(\frac{a}{r}\right)^4 \right) P_3(\cos \theta) \right. \\
\left. + \cdots \right] \]

Taking a constant $\phi$ slice of the region between the spheres, the potential looks somewhat like

We note that including the higher Legendre modes improves the potential near the surfaces of the spheres. This is much like summing the first few terms of a Fourier series. On the other hand, the potential midway between the spheres is well estimated by just the first term or two in the series. This is because both $r/b$ and $a/r$ are small in this region, and the series rapidly converges (assuming $a \ll b$, that is).
In the limit when \( b \to \infty \) we may remove \((a/b)\) and \((r/b)\) terms. Removing
the latter corresponds to having only inverse powers of \( r \) surviving, which is the
expected case for an exterior solution. The result is

\[
\Phi(r, \theta) \to \frac{V}{2} + \frac{V}{2} \left[ \frac{3}{2} \left( \frac{a}{r} \right)^2 P_1(\cos \theta) - \frac{7}{8} \left( \frac{a}{r} \right)^4 P_3(\cos \theta) + \cdots \right]
\]

which agrees with the exterior solution for a sphere with oppositely charged hemi-
spheres (except that here we have the average potential \( V/2 \) and that the potential
difference between northern and southern hemispheres is only half as large).

Similarly, when \( a \to 0 \) we remove \((a/b)\). But this time we get rid of the inverse
powers \((a/r)\) instead. The result is the interior solution

\[
\Phi(r, \theta) \to \frac{V}{2} - \frac{V}{2} \left[ \frac{3}{2} \left( \frac{r}{b} \right) P_1(\cos \theta) - \frac{7}{8} \left( \frac{r}{b} \right)^3 P_3(\cos \theta) + \cdots \right]
\]

which is again a reasonable result (this time with the hemispheres oppositely
charged from the previous case).
3.3) We would like to find the potential created by a thin, flat conducting circular disc of radius $R$ located in the $x-y$ plane, maintained at fixed potential $V$. The charge density on a disc is proportional to $1/\sqrt{R^2 - r^2}$, where $r$ is the distance from the center of the disc.

**WARNING:** Unfortunately the solution in the Jackson is wrong. Nevertheless, we will give credit to everyone who solved the problem according to Jackson!

To show that the Jackson’s solution is wrong, we note that there is no charge on the sphere at $r = R$ in 3D, hence we can use analytic continuation of $\Phi$ from $r > R$ to $r < R$ at finite $\theta$ ($\theta = 0$ might be problematic because the disc is located in the $x-y$ plane). Using analytic continuation, we conclude that Jackson proposes solution

$$\Phi(r < R) = \frac{2V}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{r}{R}\right)^{2l} P_{2l}(\cos \theta)$$  \hspace{1cm} (1)$$

which gives at $\theta = \pi/2$ non-constant potential, i.e.,

$$\Phi(r < R, \theta = \pi/2) = \frac{2V}{\pi} + \frac{2V}{\pi} \sum_{l=1}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{r}{R}\right)^{2l} P_{2l}(0) \neq V$$  \hspace{1cm} (2)

**Correct solution of 3.3**

All points in 3D, except at $(\theta = \pi/2, r < R)$, satisfy the Laplace Eq. $\nabla^2 \Phi = 0$. The solution at all points except at $\pi/2$ has to satisfy

$$\Phi_{inside}(r) = \sum_{l} B_l \left(\frac{r}{R}\right)^l P_l(\cos \theta)$$  \hspace{1cm} (3)$$

$$\Phi_{outside}(r) = \sum_{l} B_l \left(\frac{R}{r}\right)^{l+1} P_l(\cos \theta)$$  \hspace{1cm} (4)$$

We already took into account the analytic continuation across the boundary at $r = R$. We might have different $B_l$'s for $\theta < \pi/2$ and $\theta > \pi/2$, because there is a charged disc present at $\theta = \pi/2$. At $\theta = \pi/2$ the potential is continuous, but it must have discontinuous derivative (because electric field jumps). Let us concentrate on $\theta < \pi/2$ for now.

Approaching point $\pi/2$ from above, we have the condition of a constant potential on the surface of the conducting disc, i.e.,

$$V = \Phi_{inside}(r, \theta = \pi/2) = B_0 + \sum_{l=1}^{\infty} B_l \left(\frac{r}{R}\right)^l P_l(0)$$  \hspace{1cm} (5)$$

This is satisfied for all $r < R$ only if $B_0 = V$ and $B_{2n} = 0$ ($n$ integer) because $P_{2n}(0) \neq 0$. On the other hand $P_{2n+1}(0) = 0$, hence arbitrary coefficients $B_{2n+1}$ are possible. We thus have

$$\Phi_{inside}(r) = V + \sum_{n=0}^{\infty} B_{2n+1} \left(\frac{r}{R}\right)^{2n+1} P_{2n+1}(\cos \theta)$$  \hspace{1cm} (6)$$

$$\Phi_{outside}(r) = V \left(\frac{R}{r}\right) + \sum_{n=0}^{\infty} B_{2n+1} \left(\frac{R}{r}\right)^{2n+2} P_{2n+1}(\cos \theta)$$  \hspace{1cm} (7)$$

To determine coefficients $B_{2n+1}$, we note that the electric field needs to be such that it gives rise to the given surface charge distribution $\sigma = \frac{\lambda}{\sqrt{R^2 - r^2}}$. The electric field just above the conducting disc is

$$E_r = 0$$  \hspace{1cm} (8)$$

$$E_\theta = -\frac{1}{r} \frac{\partial \Phi(r, \theta = \pi/2)}{\partial \theta} = \sum_{n=0}^{\infty} B_{2n+1} \frac{r^{2n}}{R^{2n+1}} \frac{dP_{2n+1}}{dx}(0)$$  \hspace{1cm} (9)$$

The normal component of the electric field is proportional to the surface charge, hence

$$\frac{\lambda}{\sqrt{R^2 - r^2}} = \epsilon_0 E_\theta^{inside}(\theta = \pi/2),$$  \hspace{1cm} (11)$$
where we took into account that half of the charge appears on the upper part of the conducting disc, and half on the lower part. This equation has to be satisfied for every \( r < R \), hence every power of \( r \) has to match. Note that only even powers in \( r \) appear on both the right and the left hand side. We thus perform power expansion and compare term by term:

\[
\frac{\lambda}{R} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n + 1)\Gamma(1/2)} \left( \frac{r}{R} \right)^{2n} \frac{d}{dx} P_{2n+1}(0) = \sum_{n=0}^{\infty} B_{2n+1} \frac{r^{2n}}{R^{2n+1}} \frac{d}{dx} P_{2n+1}(0)
\]

(12)

We thus conclude

\[
B_{2n+1} = \frac{\lambda}{2\varepsilon_0 R^{2n+1}(0)} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n + 1)\Gamma(1/2)}
\]

(13)

which can be simplified to

\[
B_{2n+1} = \frac{\lambda (-1)^n}{2\varepsilon_0 2n + 1}
\]

(14)

We finally have for \( \theta < \pi/2 \) the following solution

\[
\Phi_{\text{inside}}(r) = V + \frac{\lambda}{2\varepsilon_0} \sum_{n=0}^{\infty} \left( \frac{-1)^n}{2n + 1} \left( \frac{r}{R} \right)^{2n+1} \frac{d}{dx} P_{2n+1}(\cos \theta)
\]

(16)

\[
\Phi_{\text{outside}}(r) = V \frac{R}{r} + \frac{\lambda}{2\varepsilon_0} \sum_{n=0}^{\infty} \left( \frac{-1)^n}{2n + 1} \left( \frac{R}{r} \right)^{2n+2} \frac{d}{dx} P_{2n+1}(\cos \theta)
\]

(17)

To continue the solution below the \( x - y \) plane (to \( \theta > \pi/2 \)) we notice that \( E_{\theta} \) must change sign across \( x - y \) plane, but it needs to be of equal strength above and below the plane. Let’s check the first few terms in \( \Phi_{\text{inside}} \) close to the \( x - y \) plane:

\[
\Phi_{\text{inside}} \approx V + \frac{\lambda}{2\varepsilon_0} \frac{z}{R} - \frac{\lambda}{12\varepsilon_0} \frac{(5z^3 - 3zr^2)}{R^3} + \cdots
\]

(18)

which gives

\[
E = e_z \left[ \frac{\lambda}{2\varepsilon_0} \frac{1}{R} + \frac{\lambda}{4\varepsilon_0} \frac{r^2}{R^3} + \cdots \right]
\]

(19)

The electric field should change sign, and therefore should be of the form

\[
E = e_z \text{sign}(z) \left[ \frac{\lambda}{2\varepsilon_0} \frac{1}{R} + \frac{\lambda}{4\varepsilon_0} \frac{r^2}{R^3} + \cdots \right]
\]

(20)

therefore potential must also contain \( \text{sign}(\cos \theta) \), i.e.,

\[
\Phi_{\text{inside}}(r) = V + \frac{\lambda}{2\varepsilon_0} \sum_{n=0}^{\infty} \left( \frac{-1)^n}{2n + 1} \left( \frac{r}{R} \right)^{2n+1} \frac{d}{dx} P_{2n+1}(\cos \theta)\text{sign}(\cos \theta)
\]

(21)

\[
\Phi_{\text{outside}}(r) = V \frac{R}{r} + \frac{\lambda}{2\varepsilon_0} \sum_{n=0}^{\infty} \left( \frac{-1)^n}{2n + 1} \left( \frac{R}{r} \right)^{2n+2} \frac{d}{dx} P_{2n+1}(\cos \theta)\text{sign}(\cos \theta)
\]

(22)

An interesting question is why does the integral of the charge not give correct answer. We would expect

\[
\Phi(r, r > R) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r')}{|r - r'|} d^3 r'
\]

(23)

Taking into account the expansion of \( \frac{1}{|r - r'|} \) in terms of spherical harmonics, and noting that the problem is independent of azimuthal angle \( \varphi \), we conclude

\[
\Phi(r, r > R) = \frac{1}{4\pi\varepsilon_0} \sum_{l} \frac{P_l(\cos \theta)}{r^{l+1}} \int \rho(r') P_l(\cos \theta') r'^l d^3 r'
\]

(24)
Naively, one would expect $\rho(r, \theta) = \frac{\lambda \delta(\cos \theta)}{r \sqrt{R^2 - r^2}}$. This charge density leads to the solution in the Jackson’s book, which is wrong. The problem is that the conducting disc does not contain only one thin layer of charge, but actually two thin layers, one above and one below, which give the charge density of the form $\rho(r, \theta) = \frac{\lambda \delta'(\cos \theta) \text{sign}(\cos \theta)}{r \sqrt{R^2 - r^2}}$, where $\delta'(x)$ is the derivative of the $\delta$-function. Only this form leads to the correct solution derived above.

The wrong solution follows
Problem 3.3

Hint: A closed expression exists for \( \int_0^R \frac{r^{2n+1}}{\sqrt{R^2 - r^2}} \, dr \) (you can find it with Mathematica, for instance).

a): Exterior potential \((r > R)\). The surface charge density is \( \sigma(r) = \frac{\lambda}{\sqrt{R^2 - r^2}} \) for \( r < R \) and zero otherwise. The volume charge density must be of the form \( \rho(x) = f(r) \delta(\cos \theta) \). The function \( f(r) \) is determined by considering the charge in a shell of radius \( r \) and thickness \( dr \):

\[
dq = \sigma(r) 2\pi r dr = \int_{\phi = 0}^{2\pi} \int_{\cos \theta = -1}^{1} d\cos \theta r^2 dr f(r) \delta(\cos \theta) = 2\pi r^2 f(r) dr \tag{28}
\]

and thus \( \rho(r) = \frac{\sigma(r)}{r} \delta(\cos \theta) \). To find \( \lambda \), we calculate the potential at the origin and equate the result to \( V \):

\[
V = \Phi(0) = \frac{1}{4\pi\varepsilon_0} \int_0^R \frac{\lambda}{r\sqrt{R^2 - r^2}} 2\pi r dr = \frac{\lambda}{2\varepsilon_0} \left[ \sin^{-1} \left( \frac{r}{R} \right) \right]_0^R = \frac{\lambda\pi}{4\varepsilon_0} \tag{29}
\]

Thus, \( \lambda = \frac{4\varepsilon_0 V}{\pi} \), and \( \rho(r) = \frac{4\varepsilon_0 V}{\pi r\sqrt{R^2 - r^2}} \delta(\cos \theta) \). Using Eq. 3.70 of the textbook for the case \( r > R \) we have \( r_+ = r \) and \( r_- = r' \), and we find

\[
\Phi(x) = \frac{1}{4\pi\varepsilon_0} \frac{4\varepsilon_0 V}{\pi} \sum_{l,m} \frac{4\pi}{2l + 1} Y_{lm}(\theta, \phi) \frac{1}{r^{l+1}} \int_0^R \int r'^2 Y_{lm}^*(\theta', \phi') \frac{1}{r'\sqrt{R^2 - r'^2}} \delta(\cos \theta') r'^2 d\cos \theta' d\phi' dr'
\]

\[
= \frac{V}{\pi^2} \sum_{l} \frac{2\pi}{r^{l+1}} P_l(\cos \theta) \int_0^R \int r'^{l+1} P_l(\cos \theta') \frac{1}{\sqrt{R^2 - r'^2}} \delta(\cos \theta') d\cos \theta' dr'
\]

\[
= \frac{2V}{\pi} \sum_{l} \frac{1}{r^{l+1}} P_l(\cos \theta) P_l(0) \int_0^R \frac{r'^{l+1}}{\sqrt{R^2 - r'^2}} dr'
\]

\[
= \frac{2V}{\pi} \sum_{n=0}^\infty \frac{1}{r^{2n+1}} P_{2n}(\cos \theta) \frac{(-1)^n(2n - 1)!!}{2^n n!} \int_0^R \frac{r'^{l+1}}{\sqrt{R^2 - r'^2}} \, dr' .
\]

Using integral tables or software, it is found that \( \int_0^R \frac{r'^{l+1}}{\sqrt{R^2 - r'^2}} \, dr' = R^{2n+1} \frac{n^{2n}}{(2n+1)!!} \), and thus

\[
\Phi_{r>R}(x) = \frac{2V}{\pi} \sum_{n=0}^\infty \frac{(-1)^n}{2n + 1} \left( \frac{R}{r} \right)^{2n+1} P_{2n}(\cos \theta) , \text{q.e.d.} \tag{30}
\]

b): Interior potential \((r < R)\). On the surface \( r = R \) the expansions for \( r > R \) and \( r < R \) must agree, i.e. the respective coefficient functions \( B_l r^{-l-1} \) and \( A_l r^l \) of the \( P_l(\cos \theta) \) must be equal for \( r = R \) and for all \( l \). Thus, the interior coefficients \( A_l = B_l R^{2l-1} \). Here, \( l = 2n \) and \( A_{2n} = B_{2n} R^{-4n-1} = R^{2n+1} R^{-4n-1} = R^{-2n} \). Thus,

\[
\Phi_{r<R}(x) = \frac{2V}{\pi} \sum_{n=0}^\infty \frac{(-1)^n}{2n + 1} \left( \frac{R}{r} \right)^{2n} P_{2n}(\cos \theta) . \tag{31}
\]

With \( r_- = \min(r, R) \) and \( r_+ = \max r, R \) the potential in all space can be written as
\[ \Phi_{\text{everywhere}}(x) = \frac{2VR}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n r_{<}^{2n}}{2n+1 2^n+1} P_{2n}(\cos \theta) . \]  

(32)

c): The capacitance is \( C = \frac{Q}{V} \). From part a), we know that \( V = \frac{3\pi \epsilon_0}{4\epsilon_0} \). The total charge \( Q \) on the disk is obtained as

\[
Q = \int_{0}^{R} \frac{\lambda}{\sqrt{R^2 - r^2}} 2\pi r dr = 2\pi \lambda \left[ -\sqrt{R^2 - r^2} \right]_{0}^{R} = 2\pi \lambda R .
\]

(33)

Thus, \( C = \frac{Q}{V} = 2\pi \lambda R \frac{4\epsilon_0}{\lambda \epsilon_0} \), and \( C = 8\epsilon_0 R \).