Midterm Exam, Quantum Mechanics 501, Rutgers

October 30, 2014

1) An electron ($m = 0.511\times 10^6$ eV/c^2) is bound in a parabolically shaped one-dimensional potential. The parameters of the potential can be found from the fact that the electron feels potential of 1eV when it is 5 Å away from the center.

   - What minimum total energy (in eV) can it have?
   - What wavelength of light (in nm) will be strongly absorbed by this electron?

   Useful constants: $\hbar c \approx 197$eV nm and mass of the electron $m = 0.511 \times 10^6$ eV/c^2

   Ans.: The frequency can be determined by $\frac{1}{2}m\omega^2 x_0^2 = V_0$, where $x_0 = 5$Å and $V_0 =$ 1eV. We get

$$\hbar\omega = \sqrt{\frac{2V_0}{mc^2}}\frac{\hbar c}{x_0} \approx 0.78\,eV \tag{1}$$

   The minimum of energy is

$$\hbar\omega/2 \approx 0.39\,\text{eV}.$$

   For the photon to be absorbed, it must have energy of the elementary excitations, $\hbar\omega$. The wavelength that corresponds to this excitaton is

$$\lambda = \frac{hc}{E} = \frac{2\pi\hbar c}{\hbar\omega} \approx 6274\,\text{nm}.$$

2) Now suppose the electron is in superposition of the first two excited states $|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$ of the Harmonic oscilator.

   - What is expectation value of the energy?
   - Find $\langle x^2\rangle - \langle x\rangle^2$ at $t = 0$ in units of $\frac{\hbar}{m\omega}$.
   - Find $\langle x\rangle(t)$ (expectation at time $t > 0$) in units of $\sqrt{\frac{\hbar}{m\omega}}$.

   Ans.:

   - The expectation value of the energy is

$$\langle E\rangle = \langle\psi|H|\psi\rangle = 2\hbar\omega.$$

     which is of course the geometric average of the two eigenenergies.

– The position operator is $\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$. The expectation values are

$$\langle X \rangle = \sqrt{\frac{\hbar}{2m\omega}} \frac{1}{2} \left(\langle 1| + \langle 2| \right) \left(a^\dagger + a \right) \left(|1\rangle + |2\rangle \right) = \sqrt{\frac{\hbar}{m\omega}} \tag{2}$$

$$\langle X^2 \rangle = \frac{\hbar}{2m\omega} \frac{1}{2} \left(\langle 1| + \langle 2| \right) \left(a^\dagger + a \right)^2 \left(|1\rangle + |2\rangle \right) = \frac{2\hbar}{m\omega} \tag{3}$$

$$\langle X^2 \rangle - \langle X \rangle^2 = \frac{\hbar}{m\omega} \tag{4}$$

The time dependent wave function is

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(e^{-iE_1 t/\hbar} |1\rangle + e^{-iE_2 t/\hbar} |2\rangle) = \frac{e^{-iE_1 t/\hbar}}{\sqrt{2}} \left(|1\rangle + e^{-i\omega t} |2\rangle \right) \tag{5}$$

hence expectation value of position is

$$\langle X \rangle (t) = \sqrt{\frac{\hbar}{2m\omega}} \frac{1}{2} \left(\langle 1| + e^{i\omega t} \langle 2| \right) \left(a^\dagger + a \right) \left(|1\rangle + e^{-i\omega t} |2\rangle \right) = \sqrt{\frac{\hbar}{m\omega}} \cos(\omega t) \tag{6}$$

3) Two quantum operators have the matrix representation

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{7}$$

a) A system is in quantum state $|\psi\rangle$ that is in an eigenfunction of operator A, corresponding to eigenvalue -1. Then for this state, what are $\langle A \rangle$ and ΔA?

b) First A is measured and the result is $a = -1$. What is the state of the system after the measurement?

c) Immediately afterwards, B is measured. What is the probability to find $b = 1$?

d) Assuming that $b = 1$ was indeed found in (c), what is the state of the system after the measurement of B?

Ans.:

a) The eigenstate has no uncertainty $\Delta A = 0$ and its average is eigenvalue $\langle A \rangle = -1$.

b) To answer the next few questions, we need to diagonalize A and B. First, notice the block-form of the matrices, hence only 2×2 diagonalization is needed. These are the eigenvalues and eigenvectors of operator A

$$\begin{array}{ll} a_0 = 1 & |a_0\rangle = \frac{1}{\sqrt{2}}(1, 1, 0) \\ a_1 = 1 & |a_1\rangle = (0, 0, 1) \\ a_2 = -1 & |a_2\rangle = \frac{1}{\sqrt{2}}(1, -1, 0) \end{array} \tag{8}$$

Similarly for operator B

$$\begin{array}{ll} b_0 = 1 & |a_0\rangle = (1, 0, 0) \\ b_1 = 1 & |a_1\rangle = \frac{1}{\sqrt{2}}(0, 1, 1) \\ b_2 = -1 & |a_2\rangle = \frac{1}{\sqrt{2}}(0, 1, -1) \end{array} \tag{9}$$

After the measurement of $a = -1$ the wave function is equal to $|a_2\rangle = \frac{1}{\sqrt{2}}(1, -1, 0)$.

- We start here with $|a_2\rangle$ and after measurement find either $|b_0\rangle$ or $|b_1\rangle$. The combined probability is

$$P(b=1) = |\langle b_0|a_2\rangle|^2 + |\langle b_1|a_2\rangle|^2 = \frac{3}{4} \tag{10}$$

- The system is left in the projected state, i.e.,

$$|\psi_{final}\rangle = \mathcal{P}|a_2\rangle = |b_0\rangle\langle b_0|a_2\rangle + |b_1\rangle\langle b_1|a_2\rangle = \frac{1}{2\sqrt{2}}(2,-1,-1)$$

  If normalized, the state becomes

$$|\psi_{final}\rangle = \frac{1}{\sqrt{6}}(2,-1,-1)$$

4) A particle in 1D is described by the usual Schroedinger equation with potential $V(x)$, which is a hybrid of the infinite well and the attractive Dirac-delta function, $V(x) = -\lambda\delta(x)$ for $|x| < L/2$ and $V(x) = +\infty$ for $|x| > L/2$. Usually we specify the parameters in H and ask for the ground state energy E_0, but this problem is backward: Assuming that the ground state energy E_0 is exactly zero, find the value of λ that makes that possible:

   a) Consider the form of the Schrodeinger equation in the region $0 < x < L/2$ and find the general form of the solution. (Don't worry if the form looks slightly surprising.)
   b) Of course, a similar form, with different coefficients, applies in $-L/2 < x < 0$. Use this knowledge to sketch the form of the wave function (which must be consistent with the boundary conditions).
   c) Find the value of λ that solves the problem.

Ans.:

a) Since there is no potential in the box (except at $x = 0$) and energy vanishes, we have

$$\begin{array}{ll} -\frac{\hbar^2}{2m}\psi'' = 0 & x < 0 \\ -\frac{\hbar^2}{2m}\psi'' = 0 & x > 0 \end{array} \tag{11}$$

   The solution is thus $\psi(x) = kx + A$

b) To satisfy the boundary condition at $x = -L/2$ and $x = L/2$ we must have

$$\psi(x) = \begin{cases} k(x + L/2) & x < 0 \\ k(L/2 - x) & x > 0 \end{cases} \tag{12}$$

c) To determine λ, we need to integrate the Schroedinger equation around the point $x = 0$, and we find

$$-\frac{\hbar^2}{2m}\int_{-\delta}^{\delta}\psi'' dx + \int_{-\delta}^{\delta}V(x)\psi(x)dx = 0 \tag{13}$$

which gives

$$\lambda = \frac{2\hbar^2}{mL}$$

Normalization will also give the coefficient $k = \sqrt{\frac{12}{L^3}}$.