Homework 7, Quantum Mechanics 501, Rutgers

December 18, 2016

1) Consider a system of two non-identical fermions, each with spin 1/2. One is in a state with $S_{1x} = \frac{\hbar}{2}$, while the other is in a state with $S_{2y} = -\frac{\hbar}{2}$. What is the probability of finding the system in a state with total spin quantum numbers $s = 1$, $m_s = 0$, where m_s refers to the z-component of the total spin?

   a) First, find the eigenstate of the operator S_{1x} with the eigenvalue $\frac{\hbar}{2}$. Also find the eigenstate of S_{2y} with the eigenvalue $-\frac{\hbar}{2}$.
   Answ.: The eigenvectors are

$$|S_{1x} = +\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$$

$$|S_{2y} = -\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - i\,|\downarrow\rangle)$$

   b) Using the rules for sumation of angular momenta, find the expression for the state $|s = 1, m_s = 0\rangle$.
   Answ.: The triplet state is

$$|s = 1, m_s = 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

   c) Calculate the probability.
   Answ.: Since the fermions are not identical, the wave function of the system is the product wave function of $|S_{1x} = +\frac{1}{2}\rangle$ and $|S_{2y} = -\frac{1}{2}\rangle$, i.e.,

$$|\psi\rangle = \frac{1}{2}(|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle - i\,|\downarrow\rangle)$$

   The probability is thus

$$P = |\,\langle s = 1, m_s = 0|\psi\rangle\,|^2 = |\frac{1}{2\sqrt{2}}(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|)(|\uparrow\uparrow\rangle + |\downarrow\uparrow\rangle - i\,|\uparrow\downarrow\rangle - i\,|\downarrow\downarrow\rangle)|^2 = |\frac{1-i}{2\sqrt{2}}|^2 = \frac{1}{4}$$

2) Consider two spin-1 particles that occupy the state

$$|s_1 = 1, m_1 = 1; s_2 = 1, m_2 = 0\rangle\,.$$

What is the probability of finding the system in an eigenstate of the total spin S^2 with quantum number $s = 1$? What is the probability for $s = 2$?

Answ.: Using Clebsh-Gordan coefficients for addition of angular momenta, we have

$$|s=1, m_s=1\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle - |0,1\rangle) \tag{1}$$

$$|s=1, m_s=0\rangle = \frac{1}{\sqrt{2}}(|1,-1\rangle - |-1,1\rangle) \tag{2}$$

$$|s=1, m_s=-1\rangle = \frac{1}{\sqrt{2}}(|0,-1\rangle - |-1,0\rangle) \tag{3}$$

On the right-hand side the notation is $|m_1, m_2\rangle$. Similarly, we can obtain

$$|s=2, m_s=2\rangle = |1,1\rangle \tag{4}$$

$$|s=2, m_s=1\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle + |1,0\rangle) \tag{5}$$

$$|s=2, m_s=0\rangle = \frac{1}{\sqrt{6}}(|1,-1\rangle + |-1,1\rangle) + \sqrt{\frac{2}{3}}|0,0\rangle \tag{6}$$

$$|s=2, m_s=-1\rangle = \frac{1}{\sqrt{2}}(|0,-1\rangle + |-1,0\rangle) \tag{7}$$

$$|s=2, m_s=-2\rangle = |-1,-1\rangle \tag{8}$$

The probabilities are then

$$P(s=1) = |\langle 1,0|s=1, ms=1\rangle|^2 + |\langle 1,0|s=1, ms=0\rangle|^2 + |\langle 1,0|s=1, ms=-1\rangle|^2 = 1/2$$

$$P(s=2) = |\langle 1,0|s=2, ms=1\rangle|^2 + |\langle 1,0|s=2, ms=0\rangle|^2 + |\langle 1,0|s=2, ms=-1\rangle|^2 = 1/2$$

3) a) Construct the spin singlet ($S = 0$) state and the spin triplet ($S = 1$) states of a two electron system.

Answ.: singlet:

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{9}$$

triplets:

$$|1,1\rangle = |\uparrow\uparrow\rangle \tag{10}$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \tag{11}$$

$$|1,-1\rangle = |\downarrow\downarrow\rangle \tag{12}$$

b) In the experiment we have two electrons, which are in the spin-singlet state. They move in the opposite direction along the y-axis, and two observers A and B measure the spin state of each electron. A measures the spin component along the z axis, and B measures the spin component along an axis making an angle

θ with the z axis in the xz-plane. Suppose that A's measurement yields a spin down state and subsequently B makes a measurement. What is the probability that B's measurement yields an up spin (measured along an axis making an angle θ with the z-axis)?

The explicit formula for the representation of the rotation operator $\exp(-i\mathbf{S}\cdot\hat{\mathbf{n}}\theta/\hbar)$ in the spin space is given by the spin 1/2 Wigner matrix

$$D^{(1/2)}(\hat{\mathbf{n}},\theta) = \begin{pmatrix} \cos(\theta/2) - in_z\sin(\theta/2) & (-in_x - n_y)\sin(\theta/2) \\ (-in_x + n_y)\sin(\theta/2) & \cos(\theta/2) + in_z\sin(\theta/2) \end{pmatrix} \tag{13}$$

and $\hat{\mathbf{n}} = n_x\vec{e}_x + n_y\vec{e}_y + n_z\vec{e}_z$ ($|\hat{\mathbf{n}}| = 1$) is the axis of rotation.

Answ.: Since the state of two electrons is singlet, and we know that the first electron points down, the second has to point up in the same coordinate system. But observer B is rotated by θ around y axis, hence we need to find how spin-up looks in the rotated coordinate system. We thus apply $D^{(1/2)}(\vec{e}_y,\theta)$ on $(1,0)$ to get

$$|\psi_B\rangle = (cos(\theta/2), sin(\theta/2)) \tag{14}$$

The probability for up-spin is thus $P(|\uparrow\rangle) = \cos^2(\theta/2)$ and for down-spin $P(|\downarrow\rangle) = \sin^2(\theta/2)$.

4) The Wigner-Eckart theorem s given by

$$\langle n'j'm'|T_q^{(l)}|njm\rangle = \langle j'm'|lq,jm\rangle\,\frac{\langle\langle n'j'|T^{(l)}|nj\rangle\rangle}{\sqrt{2j+1}} \tag{15}$$

a) Explain the meaning of the two terms on the right hand side.

Answ.: The first term is the Clebsch-Gordan coefficient, which encodes the geometric properties of the matrix element under rotation. The second is the reduced matrix element, which is a common coefficient for all m,m' quantum numbers.

b) The interaction of the electromagnetic field with a charged particle is given by

$$\Delta H = \frac{e}{2m}\mathbf{A}\cdot\mathbf{p}$$

If the electromagnetic fields are in the form of a plane wave, then $\mathbf{A} = A_0\,\hat{\varepsilon}e^{i\mathbf{kr}}$, where $\hat{\varepsilon}$ is the polarization of the plane wave. Assuming that the wavelength $\lambda = 2\pi/k$ is much larger than the atomic size, we may write

$$\mathbf{A} = A_0\hat{\varepsilon}(1 + i\mathbf{k}\cdot\mathbf{r} + \cdots)$$

such that

$$\Delta H \approx \frac{e}{2m}A_0\,\hat{\varepsilon}\cdot\mathbf{p}(1 + i\mathbf{k}\cdot\mathbf{r})$$

Here we kept both the dipole (the first term), and the quadrupole terms (the second term).

If the field is polarized along the x-axis ($\hat{\varepsilon} = \vec{e}_x$), and the wave propagation is along the z-axis ($\mathbf{k} = k\vec{e}_z$) express the Hamiltonian in terms of spherical harmonics. Note that $\mathbf{p}$ is a vector operator, and transforms under rotation as $\mathbf{r}$. For symmetry consideration you may therefore replace $\mathbf{p}$ by $C\mathbf{r}$

Answ.: The Hamiltonian for the above configuration is

$$\Delta H = \frac{e}{2m} A_0 C(x + ik\, xz) \tag{16}$$

Using the expressions for $Y'_{lm}s$ we can get

$$x = \sqrt{\frac{2\pi}{3}} r(Y_{1,-1} - Y_{1,1}) \tag{17}$$

$$xz = \sqrt{\frac{2\pi}{15}} r^2(Y_{2,-1} - Y_{2,1}) \tag{18}$$

hence

$$\Delta H = \frac{e}{2m} A_0 C \sqrt{\frac{2\pi}{3}} r(Y_{1,-1} - Y_{1,1} + i\frac{kr}{\sqrt{5}}(Y_{2,-1} - Y_{2,1})) \tag{19}$$

c) For the above configuration, derive the selection rules for the dipole and the quadrupole transitions, by considering the transition probability matrix elements $|\langle\psi_f|\Delta H|\psi_i\rangle|^2 = |\langle l_f m_f|\Delta H|l_i m_i\rangle|^2$. Note: selection rules state under which conditions is a transition possible.

Answ.: The dipole matrix elements are proportional to

$$\langle l_f m_f|\Delta H_1|l_i m_i\rangle \propto \langle l_f m_f|Y_{1,-1} - Y_{1,1}|l_i m_i\rangle \propto \langle l_f m_f|11, l_i m_i\rangle - \langle l_f m_f|1-1, l_i m_i\rangle \tag{20}$$

hence $|m_f - m_i| = 1$, and $|l_f - l_i| \leq 1$.

The quadrupole matrix elements are

$$\langle l_f m_f|\Delta H_2|l_i m_i\rangle \propto \langle l_f m_f|Y_{2,-1} - Y_{2,1}|l_i m_i\rangle \propto \langle l_f m_f|21, l_i m_i\rangle - \langle l_f m_f|2-1, l_i m_i\rangle \tag{21}$$

hence $|m_f - m_i| = 1$, and $|l_f - l_i| \leq 2$.

The explicit expressions for the spherical harmonics for $l = 1, 2$ are given by

$$Y_{1,1} = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{x+iy}{r} \qquad Y_{1,0} = \frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{z}{r} \tag{22}$$

$$Y_{2,2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\frac{(x+iy)^2}{r^2} \qquad Y_{2,1} = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\frac{(x+iy)z}{r^2} \qquad Y_{2,0} = \frac{1}{4}\sqrt{\frac{5}{\pi}}\frac{2z^2 - x^2 - y^2}{r^2} \tag{23}$$

and $Y_{l,-m} = (-1)^m Y^*_{l,m}$.