

# Homework 4, Quantum Mechanics 501, Rutgers

December 1, 2016

- 1) A particle in two dimensions is described by a wave function  $\psi(x, y)$ . We can make a variable substitution to cylindrical variables  $(x, y) \rightarrow (r, \phi)$  by defining an alternative wave function in terms of the new variables which describes the exact same state:  $\Phi(r, \phi) = \psi(x, y) = \psi(r \cos(\phi), r \sin \phi)$ . Using the definition of  $L_z$  in position  $(x, y)$  space, show that its representation in terms of these new variables is  $L_z = -i\hbar \frac{\partial}{\partial \phi}$ . Hint: Simply apply this form of  $L_z$  to  $\Phi(r, \phi)$  and show that you get the same answer as applying  $L_z$  in Cartesian coordinates to  $\psi(x, y)$ .

**Ans.:**

$$-i\hbar \frac{\partial}{\partial \phi} \Phi(r, \phi) = -i\hbar \frac{\partial}{\partial \phi} \psi(r \cos \phi, r \sin \phi) = -i\hbar \left[ -r \sin \phi \frac{\partial \psi}{\partial x} + r \cos \phi \frac{\partial \psi}{\partial y} \right] = \quad (1)$$

$$-i\hbar \left[ -y \frac{\partial \psi}{\partial x} + x \frac{\partial \psi}{\partial y} \right] = (xp_y - yp_x) \psi \quad (2)$$

- 2) A particle of mass  $\mu$  is constrained to move in a circle of fixed radius  $R$  around the origin in the  $x - y$  plane in the absence of any potential energy.
- a) Using the generalized coordinates  $q = \phi$  and  $p_\phi = L_z$ , write down the classical Hamiltonian for this system.

**Ans.:**

$$H = \frac{1}{2\mu R^2} L_z^2$$

- b) Now assume that the same system in quantum mechanics is governed by a Hamiltonian operator of the same functional form, i.e., by replacing  $L_z$  with the operator  $L_z$ . Find the eigenfunctions  $\Phi(\phi)$  and the eigenvalues  $E_i$  of this Hamiltonian. Make sure your solutions are "single-valued" (meaning for the same physical point in space, the eigenfunctions have the same value). What are allowed energies  $E_i$ ?

**Ans.:** The eigenfunctions are eigenstates of  $L_z$  operators, which we derived in the class

$$L_z \Phi(\phi) = \hbar m \Phi(\phi)$$

with

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

with  $m$  integer for the functions to be singular valued.

Eigenstates have energy

$$H\Phi(\phi) = E\Phi(\phi)$$

hence  $E = \hbar^2 m^2 / (2\mu R^2)$

- c) What can you say about the difference in energy of two eigenfunctions? What regular pattern would you observe in the frequency spectrum of photons emitted by this system when it undergoes a transition from some excited state to the ground state?

**Ans.:** The possible differences in energy are  $E_n - E_m = \hbar(n^2 - m^2)/(2\mu R^2)$ .

From the excited state to the ground state, the excitations have energy  $\hbar\omega = E_n - E_0 = \hbar^2 n^2 / (2\mu R^2)$  with  $n$  integer.

- d) Which kinematic quantities (operator expectation values) are conserved (independent of time) for this system (even when it is not in an eigenstate of the Hamiltonian)? List two examples.

**Ans.:** Any operator that commutes with the Hamiltonian will have a constant expectation value. This includes the Hamiltonian itself, the angular momentum operator  $L_z$ , its square  $L_z^2$ ,  $L^2$ , etc.

*Note: This example is somewhat problematic in that the radial part of the wave function is inconsistent with the laws of quantum mechanics. However, you can simply ignore the radial behavior and write the solutions as functions of  $\phi$  only. Keep in mind, though, that they must be single-valued for any physical space in point*

- 3) Consider an electron in the state with angular momentum  $l = 1$ . Derive the  $3 \times 3$  matrices for angular momentum operators  $L_z$ ,  $L_x$  and  $L_y$  in the space of  $|l = 1, m\rangle$  ( $m = +1, 0, -1$ ) of  $L_z$ . You can use the following well known relations:

$$L_+ = L_x + iL_y \quad (3)$$

$$L_- = L_x - iL_y \quad (4)$$

$$L_+ |1, 0\rangle = \hbar\sqrt{2} |1, 1\rangle \quad (5)$$

$$L_+ |1, -1\rangle = \hbar\sqrt{2} |1, 0\rangle \quad (6)$$

$$L_- |1, 1\rangle = \hbar\sqrt{2} |1, 0\rangle \quad (7)$$

$$L_- |1, 0\rangle = \hbar\sqrt{2} |1, -1\rangle \quad (8)$$

$L_z$  is diagonal in this basis, and takes the form

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (9)$$

Using the identities provided, we can write matrices for  $L_+$  and  $L_-$

$$L_+ = \hbar\sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (10)$$

$$L_- = \hbar\sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (11)$$

and finally for  $L_x = \frac{1}{2}(L_+ + L_-)$  and  $L_y = \frac{1}{2i}(L_+ - L_-)$

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (12)$$

$$L_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad (13)$$