Homework 2, Quantum Mechanics 501, Rutgers

September 27, 2016

1) An operator **A**, corresponding to a physical observable, has two normalized eigenstates $|\phi_1\rangle$ and $|\phi_2\rangle$ with non-degenerate eigenvalues a_1 and a_2, respectively. A second operator **B**, corresponding to a different physical observable, has normalized eigenstates $|\chi_1\rangle$ and $|\chi_2\rangle$, with eigenvalues b_1 and b_2, respectively. The two sets of eigenstates are related by

$$|\phi_1\rangle \propto 2\,|\chi_1\rangle + 3\,|\chi_2\rangle \tag{1}$$
$$|\phi_2\rangle \propto 3\,|\chi_1\rangle - 2\,|\chi_2\rangle \tag{2}$$

The physical observable corresponding to **A** is meassured and the value a_1 is obtained. Immediately afterwards, the physical observable corresponding to **B** is measured, and again immediately after that the one corresponding to **A** is remeasured. What is the probability of obtaining a_1 a second time?

Answ: First let's normalized the wave functions:

$$|\phi_1\rangle = \frac{1}{\sqrt{13}}(2\,|\chi_1\rangle + 3\,|\chi_2\rangle) \tag{3}$$
$$|\phi_2\rangle = \frac{1}{\sqrt{13}}(3\,|\chi_1\rangle - 2\,|\chi_2\rangle) \tag{4}$$

After the first measurement of **A**, we know that the state must have "collapsed" into the eigenstate $|\phi_1\rangle$. If we now measure **B**, the result can be either b_1 or b_2 with the following probability

$$P(b_1) = |\,\langle\chi_1|\phi_1\rangle\,|^2 \tag{5}$$
$$P(b_2) = |\,\langle\chi_2|\phi_1\rangle\,|^2 \tag{6}$$

The wave function will now be either $|\chi_1\rangle$ or $|\chi_2\rangle$, depending on whether we got b_1 or b_2. If we now measure **A** again, the probability to measure a_1 is given by

$$P(a_1) = |\,\langle\phi_1|\chi_1\rangle\,|^2 P(b_1) + |\,\langle\phi_1|\chi_2\rangle\,|^2 P(b_2) = |\,\langle\phi_1|\chi_1\rangle\,|^4 + |\,\langle\phi_1|\chi_2\rangle\,|^4 = \frac{97}{169}. \tag{7}$$

2) The ammonia molecule NH_3 has two different possible configurations: One (which we will call $|1\rangle$), where the nitrogen atom is located above the plane spanned by the three

H atoms, and the other one (which we will call $|2\rangle$) where it is below. These two states span the Hilbert space in our simple example. In both states, the expectation value of the energy $\langle n| H |n\rangle$ is the same, $E(n = 1, 2)$. On the other hand, the two states are not eigenstates of the Hamiltonian; in fact, we have $\langle 2| H |1\rangle = \langle 1| H |2\rangle = -V$ (where V is some positive number).

1) Write down the Hamiltonian in Dirac form and in matrix form.
   Answ: The Dirac form is
   $$H = E(|1\rangle\langle 1| + |2\rangle\langle 2|) - V(|2\rangle\langle 1| + |1\rangle\langle 2|). \tag{8}$$
   and the matrix form is
   $$H = \begin{pmatrix} E & -V \\ -V & E \end{pmatrix} \tag{9}$$
2) Find both eigenvalues and normalized eigenvectors. Which state is the ground state?
   Answ: The eigenvalues are $\varepsilon_{1,2} = E \pm V$.
   The ground state has energy $\varepsilon_1 = E - V$ and eigenstate $|\phi_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$. The excited state energy is $\varepsilon_2 = E + V$ with eigenstate $|\phi_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$.
3) Where is nitrogen in the two eigenstates, i.e., what is the probability to find nitrogen atom above or below in the two states?
   Answ: These two states are linear superpositions of the two eigenstates to the operator measuring the position of the N atom. In fact, it is clear that for both of them, the probability to find the N either above or below the H_3 plane is 50%. Nitrogen is in both positions at once.
4) Consider the parity operator in which all coordinates change sign ($x \to -x$). Is parity well defined in the two eigenstates? If yes, what is the value of the parity operator in the two cases?
   Answ: Under the parity operation, all coordinates change sign, interchanging the meaning of up and down. This means that $|1\rangle \to |2\rangle$ and $|2\rangle \to |1\rangle$ under this operation.
   Applying this to the two energy eigenstates, we find that the ground state is invariant (it is an eigenstate of the parity operator with eigenvalue $+1$ even parity) and the excited state acquires a minus sign (it is also an eigenstate under parity, but with eigenvalue -1, i.e., odd parity).

3) Consider the following three operators (representing physical observable) on the two dimensional Hilbert space:
$$S_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{10}$$
$$S_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{11}$$
$$S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{12}$$

(1) Assume S_z is measured and one finds the value -1. Immediately afterwards, what are $\langle S_x \rangle$, $\langle S_x^2 \rangle$ and $\Delta S_x \equiv \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2}$.

Answ.: After measurement, the system is in the eigenstate $|\psi\rangle = (0, 1)$ of S_z. Hence, we can calculate

$$\langle S_x \rangle = \langle \psi | S_x | \psi \rangle = 0 \tag{13}$$
$$\langle S_x^2 \rangle = \langle \psi | S_x^2 | \psi \rangle = 1 \tag{14}$$
$$\Delta S_x = 1. \tag{15}$$

(2) What are the possible values one could measure for S_x, and what are their possibilities if it is measured immediately after measurement in (1).

Answ. : Eigenstates of S_x are

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(1, 1) \tag{16}$$
$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(1, -1) \tag{17}$$

The first with eigenvalue $s_x = 1$ and the second with $s_x = -1$. Hence, one could measure either $+1$ or -1.

We measure $s_x = 1$ with probability $P(1) = |\langle \phi_1 | \psi \rangle|^2 = 1/2$ and $P(-1) = |\langle \phi_2 | \psi \rangle|^2 = 1/2$.

(3) Explicitely calculate the commutators between any two of the three operators above (all 3). Is it possible to prepare a state of the system with well-defined values for all three?

Answ.: After some algebra, we get

$$[S_x, S_y] = S_x S_y - S_y S_x = 2i S_z \tag{18}$$
$$[S_y, S_z] = 2i S_x \tag{19}$$
$$[S_z, S_x] = 2i S_y \tag{20}$$

Since these operators do not commute, it is not possible to find simultaneous eigenvectors to any two of them.

4) Consider the following Hamiltonian for a classical system:

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x^2 + y^2 + z^2) \tag{21}$$

Prove that the angular momentum is a constant of motion by explicitly evaluating Poisson bracket of say $L_z = xp_y - yp_x$ and H. Note that for such classical system $dL_z/dt = \{L_z, H\}$.

Ans.:

We just need to work out the possession bracket

$$\{L_z, H\} = \frac{\partial L_z}{\partial x}\frac{\partial H}{\partial p_x} + \frac{\partial L_z}{\partial y}\frac{\partial H}{\partial p_y} + \frac{\partial L_z}{\partial z}\frac{\partial H}{\partial p_z} - \frac{\partial L_z}{\partial p_x}\frac{\partial H}{\partial x} - \frac{\partial L_z}{\partial p_y}\frac{\partial H}{\partial y} - \frac{\partial L_z}{\partial p_z}\frac{\partial H}{\partial z} \tag{22}$$

Explicit evaluation gives

$$\{L_z, H\} = y\frac{\partial H}{\partial x} - x\frac{\partial H}{\partial y} = y\frac{dV}{dr^2}\frac{\partial(x^2+y^2+z^2)}{\partial x} - x\frac{dV}{dr^2}\frac{\partial(x^2+y^2+z^2)}{\partial y} = 2(xy - yx)\frac{dV}{dr^2} = 0 \quad (23)$$

which proves that L_z is a constant of motion.