Solutions to Homework 1, Quantum Mechanics 501, Rutgers

September 18, 2016

1) Prove Schwartz inequality:

$$| \langle v|w\rangle | \leq |v||w| \tag{1}$$

and triangle inequality

$$|v + w| \leq |v| + |w| \tag{2}$$

Answ.: One way to prove it is to first construct an arbitrary vector

$$|z\rangle = |v\rangle - |w\rangle \langle w|v\rangle \frac{1}{|w|^2} \tag{3}$$

and then use the basic property of scalar product $\langle z|z\rangle \geq 0$.

$$\langle z|z\rangle = \left(\langle v| - \langle w| \langle v|w\rangle \frac{1}{|w|^2}\right) \left(|v\rangle - |w\rangle \langle w|v\rangle \frac{1}{|w|^2}\right) \tag{4}$$

$$= |v|^2 - 2 \langle w|v\rangle \langle v|w\rangle \frac{1}{|w|^2} + \langle v|w\rangle \langle w|v\rangle \frac{1}{|w|^2} \tag{5}$$

hence

$$|v|^2 \geq \langle v|w\rangle \langle w|v\rangle \frac{1}{|w|^2} \tag{6}$$

$$|v|^2|w|^2 \geq | \langle v|w\rangle |^2 \tag{7}$$

which proofs the Schwartz inequality.

To prove triangle inequality, we write

$$|v + w|^2 = \langle v + w|v + w\rangle = \langle v|v\rangle + \langle w|w\rangle + 2\Re(\langle v|w\rangle) \leq |v|^2 + |w|^2 + 2| \langle v|w\rangle |. \tag{8}$$

Now using Schwartz, we know

$$|v + w|^2 \leq |v|^2 + |w|^2 + 2| \langle v|w\rangle | \leq |v|^2 + |w|^2 + 2|v||w| = (|v| + |w|)^2 \tag{9}$$

which proves triangle inequality.

2) a) Do functions defined on the interval $[0...L]$ and that vanish at the end points $x = 0$ and $x = L$ form a vector space?
Answ: Yes.

b) How about periodic functions obeying $f(L) = f(0)$?
Answ: Yes. Periodic functions form a vector space. (It may be impossible, though, to introduce a workable inner product).

c) How about all functions with $f(0) = 4$?
Answ: No. This vector space wouldnt behave properly under addition: $(f+g)(x) = f(x) + g(x)$ wouldnt work for $(4 + 4 \neq 4)$.

3) Consider the vector space **V** spanned by real 2×2 matrices.

a) What is its dimension?
Answ: 4 dimensional

b) What would be a suitable basis?
Answ:

$$|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; |2\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; |3\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; |4\rangle = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \tag{10}$$

c) Consider three example vectors from this space:

$$|1\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; |2\rangle = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; |3\rangle = \begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix} \tag{11}$$

Are they linearly independent? Support your answer with details.
Answ: The three vectors given are not linearly independent, since $|1\rangle - 2\,|2\rangle = |3\rangle$.

4) Consider the two vectors $\vec{A} = 3\hat{\imath} + 4\hat{\jmath}$ and $\vec{B} = 2\hat{\imath} - 6\hat{\jmath}$ in the 2-dimensional space of the x-y plane. Do they form a suitable set of basis vectors? (Explain.) Do they form an orthonormal basis set? If not, use Gram-Schmidt algorithm to turn them into an othomormal set.

Answ: The 2 vectors are linearly independent. Since the space has only 2 dimensions, they therefore form a basis. However, they are neither normalized nor orthogonal to each other. To turn them into an orthonormal set, first we have to normalize the first one: $\hat{A} = 0.6\hat{\imath} + 0.8\hat{\jmath}$. Then, we determine the orthogonal part of the 2nd vector: $\vec{B}' = 4.16\hat{\imath} - 3.12\hat{\jmath}$. Finally, we normalize $\vec{B}'$ to obtain $\hat{B} = 0.8\hat{\imath} - 0.6\hat{\jmath}$.

5) Assume the two operators Ω and Λ are Hermitian. What can you say about

a) $\Omega\Lambda$
Answ: The product is not necessary Hermitian, because $(\Omega\Lambda)^\dagger = \Lambda\Omega$, which is not equal to $\Omega\Lambda$, unless the two matrices commute.

b) $\Omega\Lambda + \Lambda\Omega$
Answ: The anticommutator is Hermitian.

c) $[\Omega, \Lambda]$
Answ: The commutator is antihermitian, i.e. $(\Omega\Lambda - \Lambda\Omega)^\dagger = -(\Omega\Lambda - \Lambda\Omega)$

6) Consider the matrix

$$\Omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{12}$$

a) Is it Hermitian?
Answ: Yes

b) Find its eigenvalues and eigenvectors.
Answ: The eigenvalues are +1, 0, -1. The corresponding normalized eigenvectors are

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{13}$$

c) Verify that $U^\dagger \Omega U$ is diagonal, U being the matrix formed by using each normalized eigenvector as one of its columns. (Show that U is unitary!)

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = U^\dagger \tag{14}$$

and

$$U^\dagger \Omega U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{15}$$

d) Calculate $\exp(i\Omega)$ and show it is unitary.
Answ:

$$\Lambda \equiv \exp(i\Omega) = \begin{pmatrix} \cos(1) & 0 & i\sin(1) \\ 0 & 1 & 0 \\ i\sin(1) & 0 & \cos(1) \end{pmatrix} \tag{16}$$

which can be verified by direct multiplication that

$$\Lambda\Lambda^\dagger = 1 \tag{17}$$

7) Consider the "Theta-funcion"

$$\theta(x-x') = \begin{cases} 1 & x \geq x' \\ 0 & otherwise \end{cases}$$

Show that $\delta(x-x') = \frac{d\theta(x-x')}{dx}$ by multiplaying on the r.h.s with an arbitrary square-integrable function $f(x)$ and integrating over all x.

Answ: The left side gives:

$$\int \delta(x-x')f(x')dx' = f(x) \tag{18}$$

The right side gives

$$\int \frac{d\theta(x-x')}{dx} f(x')dx' = \frac{d}{dx}\int^{x} f(x')dx' = f(x) \tag{19}$$

hence it is equal for arbitrary $f(x)$.

8) Consider a ket space spanned by the eigenkets $\{|a_i\rangle\}$ and eigenvalues $\{a_i\}$ of a Hermitian operator **A** of dimension n. There is no degeneracy.

a) Prove that operator

$$\prod_{i=1}^{n}(A-a_i)$$

is a null operator $|0\rangle$ in this space.
Answ: Assume that $|\alpha\rangle$ is an arbitrary vector in this space. The action of operator gives

$$\prod_{i=1}^{n}(A-a_i)\,|\alpha\rangle = \sum_{j=1}^{n}\prod_{i=1}^{n}(A-a_i)\,|a_j\rangle\,\langle a_j|\,|\alpha\rangle \tag{20}$$

where we inserted identity in the considered space. This can be rearanged to

$$\sum_{j=1}^{n}\langle a_j|\,|\alpha\rangle\prod_{i=1}^{n}(a_j-a_i)\,|a_j\rangle \tag{21}$$

because $|a_j\rangle$ are eigenvectors of operator A. For each term in the sum over j, the product runs over all eigenvalues $i = 1...n$, and hence the term $(a_j - a_j)$ is contained in the product, which makes the whole product to vanish. We therefore have

$$\sum_{j=1}^{n}\langle a_j|\,|\alpha\rangle\,0\times\prod_{i\neq j}^{n}(a_j-a_i)\,|a_j\rangle = |0\rangle \tag{22}$$

b) What type of projector is this operator

$$\prod_{j=1, j\neq i}^{n} \frac{1}{a_i - a_j}(A - a_j) \quad ?$$

Answ: We again start with expanding over eigenvectors

$$\prod_{j\neq i}^{n} \frac{1}{a_i - a_j}(A - a_j)\left|\alpha\right\rangle = \sum_{k=1}^{n}\prod_{j\neq i}^{n} \frac{1}{a_i - a_j}(A - a_j)\left|a_k\right\rangle\left\langle a_k\right|\left|\alpha\right\rangle = \tag{23}$$

$$\sum_{k=1}^{n}\left\langle a_k\right|\left|\alpha\right\rangle\prod_{j\neq i}^{n} \frac{1}{a_i - a_j}(a_k - a_j)\left|a_k\right\rangle \tag{24}$$

The only time this product does not vanish is when $k = i$, because j is then never equal to k. In the case of $k = i$ we have

$$\left\langle a_i\right|\left|\alpha\right\rangle\prod_{j\neq i}^{n} \frac{1}{a_i - a_j}(a_i - a_j)\left|a_i\right\rangle = \left|a_i\right\rangle\left\langle a_i\right|\left|\alpha\right\rangle \tag{25}$$

Hence the above defined operator is projector to eigenvector $\left|a_i\right\rangle$, i.e., $\left|a_i\right\rangle\left\langle a_i\right|$.