Final Exam, Quantum Mechanics 501, Rutgers

December 19, 2014

1) This problem concerns Clebsch-Gordan coefficients $\langle jm|j_1m_1, j_2m_2\rangle$.

   a) What are allowed values of total j for the addition of angular momenta $j_1 = 3$ and $j_2 = 1$?

   Answ.: 2, 3, 4

   b) Explain why $\langle 44|33, 11\rangle = 1$.

   Answ.: This is the state with maximum m, which is $m = j$. There is only one way to get $|j, m\rangle = |4, 4\rangle$, namely with maximum $m_1 = 3$ and maximum $m_2 = 1$.

   c) Find $\langle 43|32, 11\rangle$ (Hint: Use the spin lowering operator).

   Answ.: First apply total J_-

$$J_- |44\rangle = \hbar\sqrt{8}\,|43\rangle \, .$$

   The same operator $J_- = J_{1-} + J_{2-}$ can be applied to $|33, 11\rangle$ to obtain

$$(J_{1-} + J_{2-})\,|33, 11\rangle = \hbar\sqrt{6}\,|32, 11\rangle + \hbar\sqrt{2}\,|33, 10\rangle \, .$$

   The resulting states are the same, from which we get

$$|43\rangle = \frac{\sqrt{3}}{2}\,|32, 11\rangle + \frac{1}{2}\,|33, 10\rangle \tag{1}$$

   and finally

$$\langle 43|32, 11\rangle = \frac{\sqrt{3}}{2}$$

2) A system is in a state described by the wavefunction $\psi(\mathbf{r}) = f(r)(x + iy + z)$, where $f(r)$ is a radial wave function. If L_z is measured, what are the possible values of the measurement, and their probabilities?

   Note that $Y_{00} = \sqrt{\frac{1}{4\pi}}$, $Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$ and $Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$.

   Answ.: The wave function can be written as

$$\psi(r) = rf(r)\sqrt{\frac{4\pi}{3}}[-\sqrt{2}\,Y_{11} + Y_{10}] \tag{2}$$

The measurement of L_z can yield $m = 1$ or $m = 0$. The probabilities are

$$P(m=0) = \frac{|\langle 10|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{\int |Y_{1,0}|^2 d\Omega}{\int |Y_{10} - \sqrt{2}Y_{11}|^2 d\Omega} = \frac{1}{\int [|Y_{10}|^2 + 2|Y_{11}|^2] d\Omega} = \frac{1}{3} \quad (3)$$

$$P(m=1) = \frac{|\langle 11|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{2\int |Y_{1,1}|^2 d\Omega}{\int |Y_{10} - \sqrt{2}Y_{11}|^2 d\Omega} = \frac{2}{\int [|Y_{10}|^2 + 2|Y_{11}|^2] d\Omega} = \frac{2}{3} \quad (4)$$

3) Consider a system of two non-identical fermions, each with spin 1/2. One is in a state with $S_{1y} = \frac{\hbar}{2}$, while the other is in a state with $S_{2x} = -\frac{\hbar}{2}$. What is the probability of finding the system in a state with total spin quantum numbers $s = 0$?

Answ.: The relevant eigenvectors of S_x and S_y are:

$$|S_x = -\frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle) \quad (5)$$

$$|S_y = \frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + i\,|\downarrow\rangle) \quad (6)$$

The two particles are non-identical, hence the Q.M. state is direct product

$$|\psi\rangle = |S_y = \frac{\hbar}{2}\rangle \otimes |S_x = -\frac{\hbar}{2}\rangle = \frac{1}{2}(|\uparrow\rangle + i\,|\downarrow\rangle) \otimes (|\uparrow\rangle - |\downarrow\rangle) = \frac{1}{2}(|\uparrow\uparrow\rangle + i\,|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle - i\,|\downarrow\downarrow\rangle) \quad (7)$$

The singlet state is $|s = 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

The probability is

$$P = |\langle s = 0|\psi\rangle|^2 = \frac{|1 - i|^2}{8} = \frac{1}{4} \quad (8)$$

4) A particle of reduced mass $\mu = 470\, MeV/c^2$ is moving in a spherical potential well of range a and depth $V_0 = -76.73\, MeV$. $[V(\mathbf{r}) = V_0$ for $|\mathbf{r}| < a$ and $V(\mathbf{r}) = 0$ for $|\mathbf{r}| > a]$.

The particle is bound in the $1s$ ground state with binding energy $E = -2.225\, MeV$. (This is supposed to be a very simple model of the deuteron). Note: $\hbar c = 197.327\, MeV fm$.

a) Solve the Schroedinger equation for both $r < a$ and for $r > a$.

b) Using the boundary conditions at $r = a$, extract the size of the "potential range" a.

c) Calculate the probability that a measurement of r will find $r > a$, i.e. the particle is outside the range of the potential (which is of course forbidden classically).

Answ.: The radial wave function for $l = 0$ solution is

$$\psi(r < a) = A\frac{\sin(kr)}{r} \quad (9)$$

$$\psi(r > a) = C\frac{e^{-\kappa r}}{r} \quad (10)$$

where

$$k = \sqrt{\frac{2\mu(E - V_0)}{\hbar^2}}$$

and

$$\kappa = \frac{2\mu|E|}{\hbar^2}$$

Given the numbers in the text, we can get

$$k = 1.341/fm$$

$$\kappa = 0.232/fm$$

The continuity of the wave function and its derivative at $r = a$ gives the following set of equations

$$A \sin(ka) = Ce^{-\kappa a} \tag{11}$$

$$A\, k \cos(ka) = -C\kappa e^{-\kappa a} \tag{12}$$

which is satisfied if

$$\tan(ka) = -\frac{k}{\kappa}. \tag{13}$$

This equation can be solved for range parameter a, and the first solution ($1s$) gives:

$$a = \frac{1}{k}(\pi - \arctan(k/\kappa)) \approx 1.3fm$$

The probability for the particle to be outside the well is

$$P(r > a) = \frac{\int_a^\infty |\psi(r)|^2 r^2 dr}{\int_0^\infty |\psi(r)|^2 r^2 dr} \tag{14}$$

The integration inside the well gives $A^2 \int_0^a \sin^2(kr)dr = A^2 \frac{a}{2}(1 - \frac{\sin(2ka)}{2ka})$ and integration outside the box gives $C^2 \int_a^\infty e^{-2\kappa r} dr = C^2 \frac{e^{-2\kappa a}}{2\kappa}$. We also have $C/A = e^{\kappa a} \sin(ka)$

The ratio that describes the probability $P(r > a)$ is

$$\frac{\sin^2(ka)}{\sin^2(ka) + \kappa a(1 - \frac{\sin(2ka)}{2ka})} \approx 0.75 \tag{15}$$

5) Two elementary particles of spin s_1 and s_2 are bound by an attractive spin-dependent potential, as specified by the Hamiltonian

$$H = \frac{p^2}{2\mu} + U(r) + V(r)\mathbf{S}_1 \cdot \mathbf{S}_2 \tag{16}$$

where $\mathbf{r}$ and $\mathbf{p}$ are relative coordinate and momentum; μ is the reduced mass; $U(r)$ and $V(r)$ are two different spherically symmetric potentials; and $\mathbf{S}_1$ and $\mathbf{S}_2$ are the spin operators for two particles. (We ignore the center-of-mass motion).

a) The Hamiltonian can also be written as

$$H = [\frac{p^2}{2\mu} + U]^{(c)} \otimes I^{(s)} + V^{(c)} \otimes [\frac{1}{2}(S^2 - S_1^2 - S_2^2)]^{(s)} \tag{17}$$

Briefly explain the notation used above, explain why certain terms appear before or after the '$\otimes$' and show how the last terms involving spins was obtained, in which $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$

Answ.: We separated the Hilbert space corresponding to the real space coordinates (c) from the spin space part (s). The two can be written as direct product because they correspond to different Hilbert spaces.

The last term was simplified by the use of the identity $\mathbf{S}^2 = \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2\mathbf{S}_1\mathbf{S}_2$

b) Show that a vector of the form $|\psi_{nsm}\rangle = |\chi_{ns}\rangle \otimes |sms_1s_2\rangle$ is an eigenvector of H if $|\chi_{ns}\rangle$ obeys the effective Schroedinger equation

$$[-\frac{\hbar^2\nabla^2}{2\mu} + U + VC_s]\,|\chi_{ns}\rangle = E\,|\chi_{ns}\rangle \tag{18}$$

with $C_s = (\hbar^2/2)[s(s+1) - s_1(s_1+1) - s_2(s_2+1)]$. Here the state $|sms_1s_2\rangle$ is bulit from states $|s_1m_1, s_2m_2\rangle$ according to the usual rules for addition of angular momenta.

Answ.: The action of the first term on the wave function $|\psi_{nsm}\rangle$ gives

$$\left([\frac{p^2}{2\mu} + U]\,|\chi_{ns}\rangle\right) \otimes |sms_1s_2\rangle \tag{19}$$

The second term evaluates to

$$(V\,|\chi_{ns}\rangle) \otimes \left(\frac{1}{2}(S^2 - S_1^2 - S_2^2)\,|sms_1s_2\rangle\right) \tag{20}$$

and since the spin operator on the eigenstate of s, s_1, s_2 gives an eigenvalue $\hbar^2(s(s+1) - s_1(s_1+1) - s_2(s_2+1))$, we can simplify the second term to

$$\frac{\hbar^2}{2}\left(s(s+1) - s_1(s_1+1) - s_2(s_2+1)\right)(V\,|\chi_{ns}\rangle) \otimes |sms_1s_2\rangle \tag{21}$$

We can put them together to obtain

$$H\,|\psi_{nsm}\rangle = [\frac{p^2}{2\mu} + U]\,|\chi_{ns}\rangle \otimes |sms_1s_2\rangle + C_sV\,|\chi_{ns}\rangle \otimes |sms_1s_2\rangle \tag{22}$$

hence Schroedinger equation is satisfied if

$$\langle\mathbf{r}|H|\psi_{nsm}\rangle = [-\frac{\hbar^2\nabla^2}{2\mu} + U(\mathbf{r}) + C_sV(\mathbf{r})]\chi_{ns}(\mathbf{r}) \otimes |sms_1s_2\rangle = E\,\langle\mathbf{r}|\psi_{nsm}\rangle = E\chi_{ns}(\mathbf{r}) \otimes |sms_1s_2\rangle \tag{23}$$

We can drop $|sms_1s_2\rangle$ and get the desired Schroedinger equation for $\chi(\mathbf{r})$.